CS103
Winter 2025


Lecture 26:

# Complexity Theory

## Part 2 of 2

# Recap from Last Time

# The Complexity Class **P**

- The complexity class **P** (***polynomial time***) is defined as

  **P** = { $L$ | There is a polynomial-time decider for $L$ }

- Intuitively, **P** contains all decision problems that can be solved efficiently.
- This is like class **R**, except with "efficiently" tacked onto the end.

# The Complexity Class **NP**

- The complexity class **NP** (***nondeterministic polynomial time***) contains all problems that can be verified in polynomial time.
- Formally:

$$\mathbf{NP} = \{ L \mid \text{There is a polynomial-time verifier for } L \}$$

- Intuitively, **NP** is the set of problems where "yes" answers can be checked efficiently.
- This is like the class **RE**, but with "efficiently" tacked on to the definition.

The Biggest Unsolved Problem in Theoretical Computer Science:

$$\mathbf{P \overset{?}{=} NP}$$

$$\mathbf{P} = \{ L \mid \text{there is a polynomial-time decider for } L \}$$

$$\mathbf{NP} = \{ L \mid \text{there is a polynomial-time verifier for } L \}$$

$$\mathbf{R} = \{ L \mid \text{there is a } \sout{\text{polynomial-time}} \text{ decider for } L \}$$

$$\mathbf{RE} = \{ L \mid \text{there is a } \sout{\text{polynomial-time}} \text{ verifier for } L \}$$

We know that **R ≠ RE**.

So does that mean **P ≠ NP**?

# A Problem

- The **R** and **RE** languages correspond to problems that can be decided and verified, *period*, without any time bounds.
- To reason about what's in **R** and what's in **RE**, we used two key techniques:
  - ***Universality***: TMs can simulate other TMs.
  - ***Self-Reference***: TMs can get their own source code.
- Why can't we just do that for **P** and **NP**?

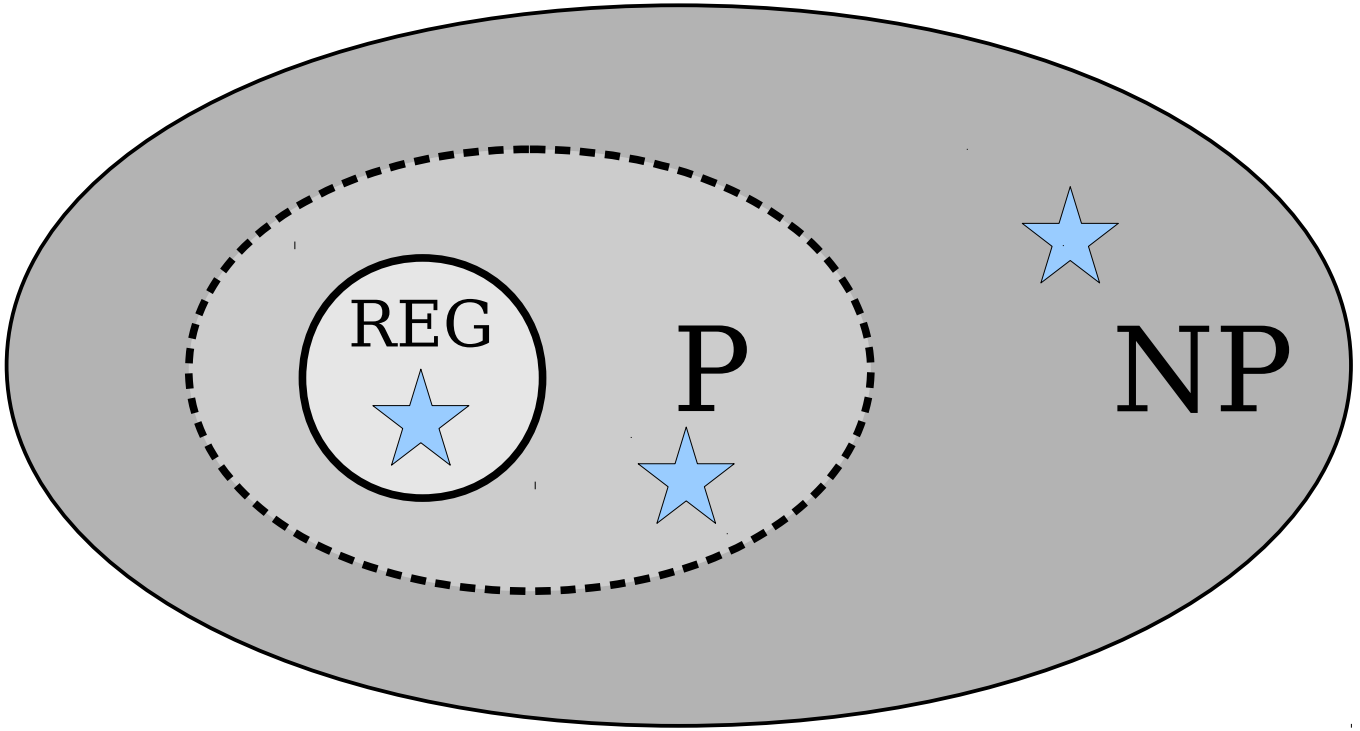
***Theorem (Baker-Gill-Solovay):*** Any proof that purely relies on universality and self-reference cannot resolve $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$.

***Proof:*** Take CS154!

So how *are* we going to reason about **P** and **NP**?

New Stuff!

# A Challenge




Problems in **NP** vary widely in their difficulty, even if **P** = **NP**.

How can we rank the relative difficulties of problems?

# Reducibility

# Maximum Matching

- Given an undirected graph $G$, a ***matching*** in $G$ is a set of edges such that no two edges share an endpoint.
- A ***maximum matching*** is a matching with the largest number of edges.

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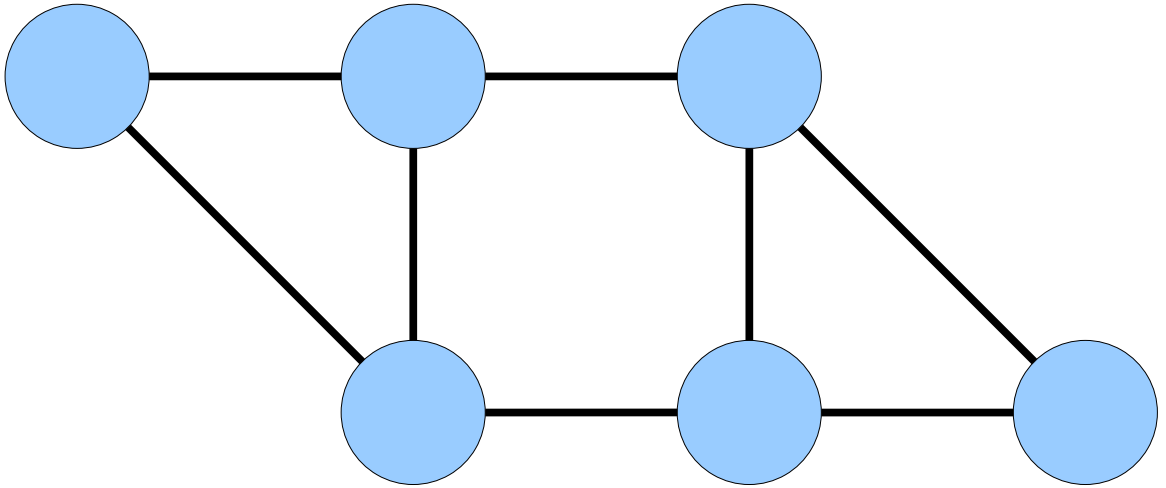

# Maximum Matching

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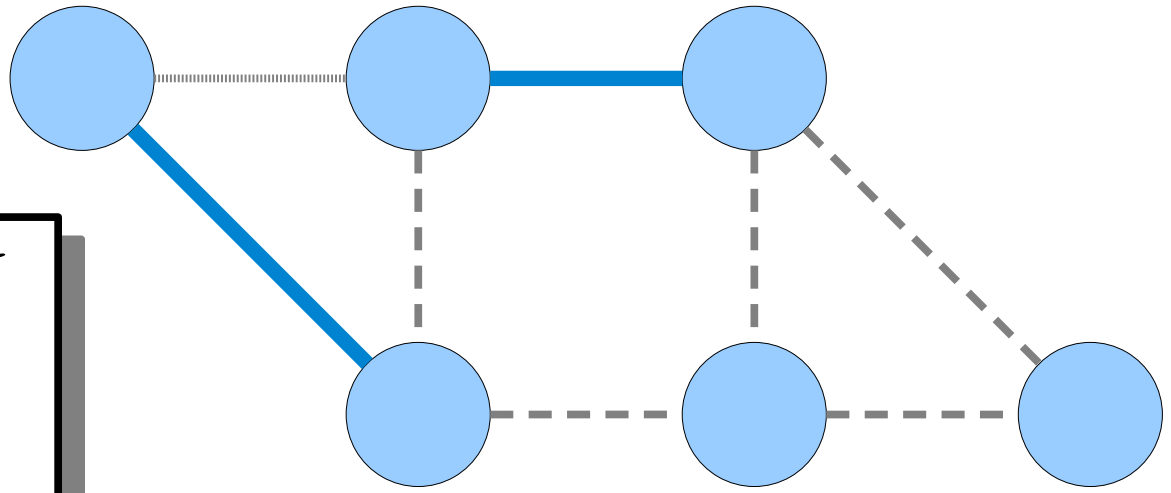

A matching, but not a maximum matching.

# Maximum Matching

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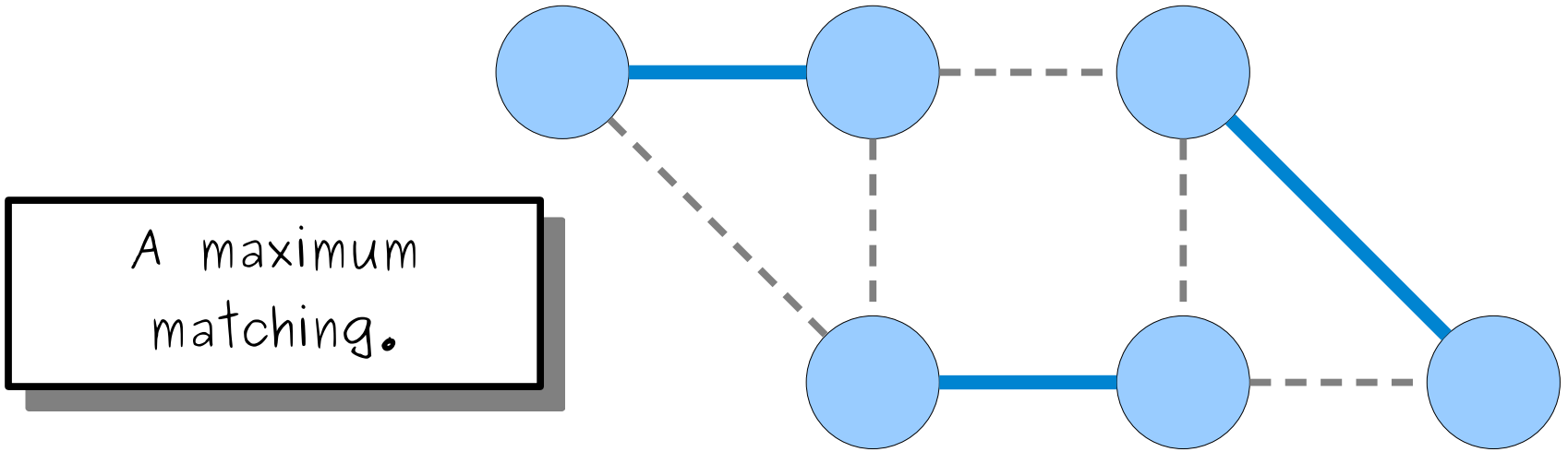
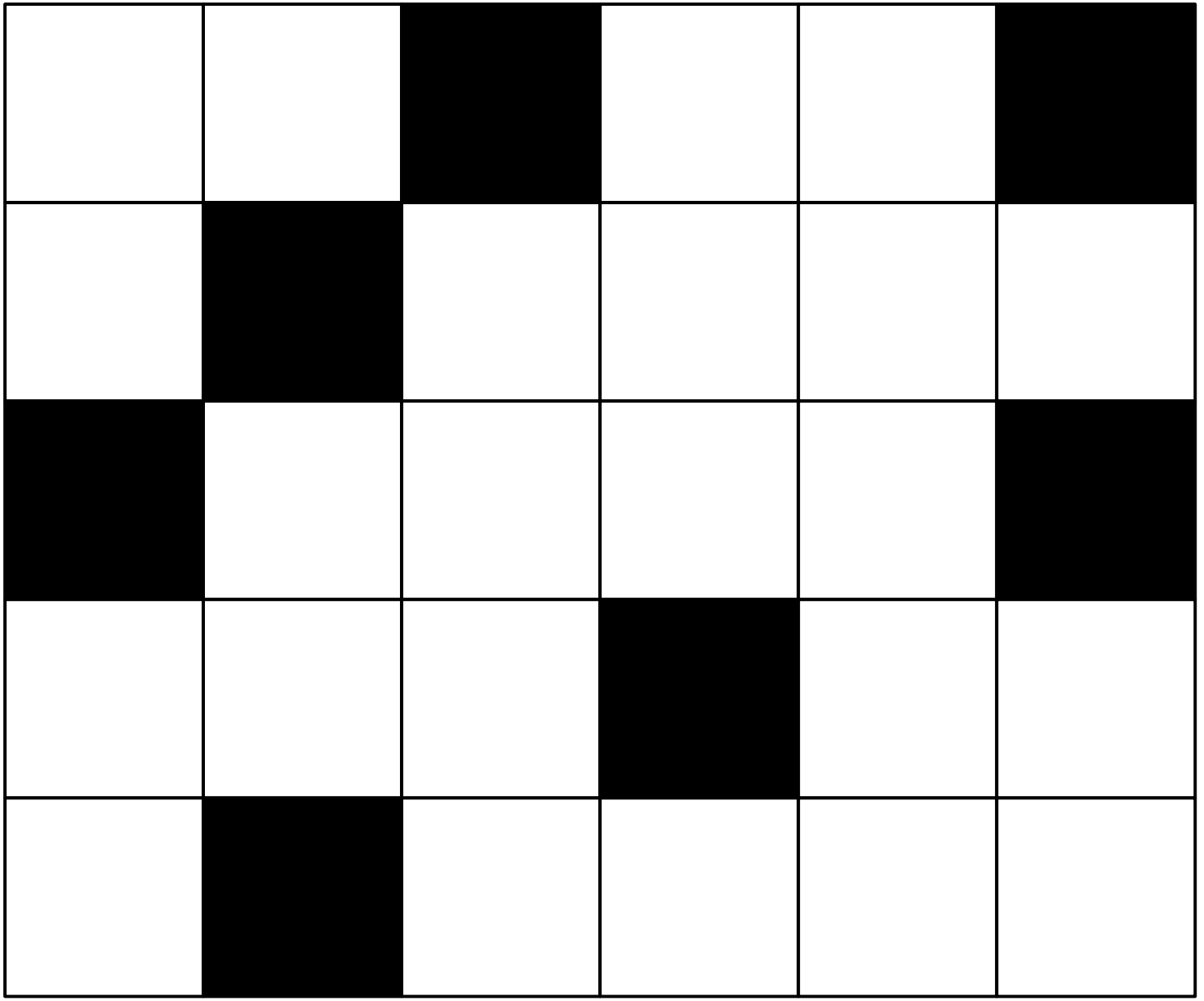


# Maximum Matching

- Jack Edmonds' paper “Paths, Trees, and Flowers” gives a polynomial-time algorithm for finding maximum matchings.
  - He’s the guy from last time with the quote about “better than decidable.”
- Using this fact, what other problems can we solve?

# Domino Tiling



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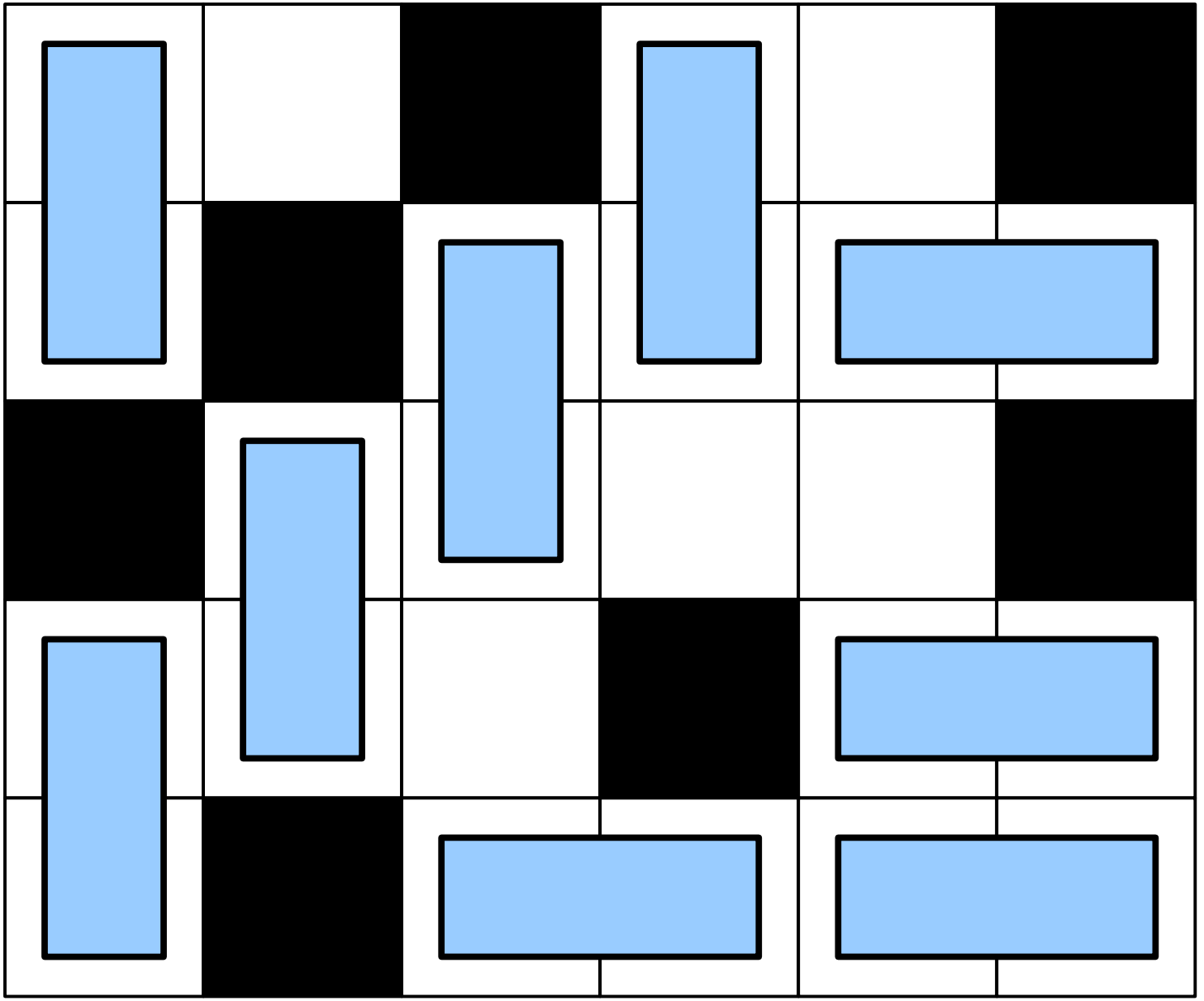

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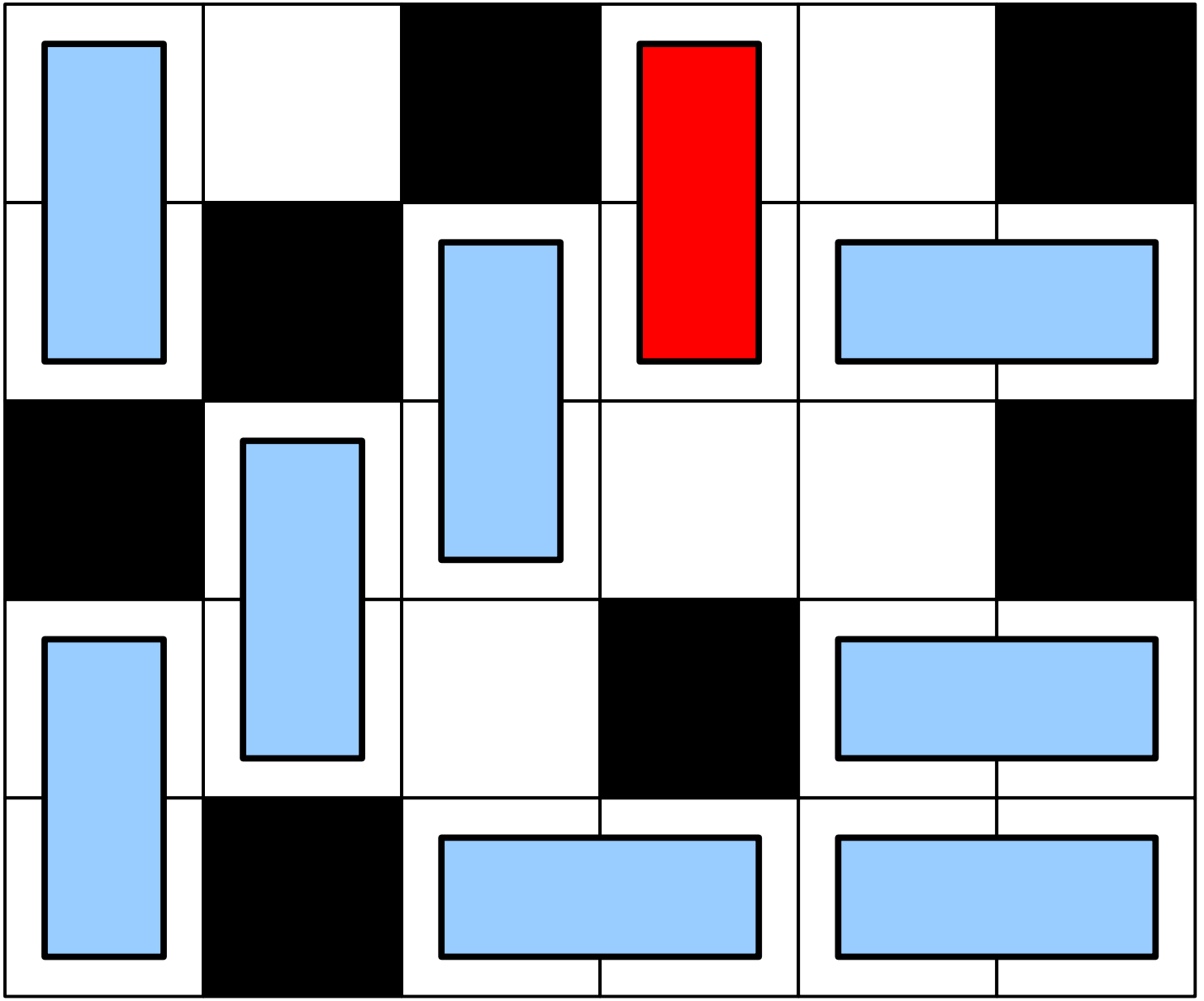

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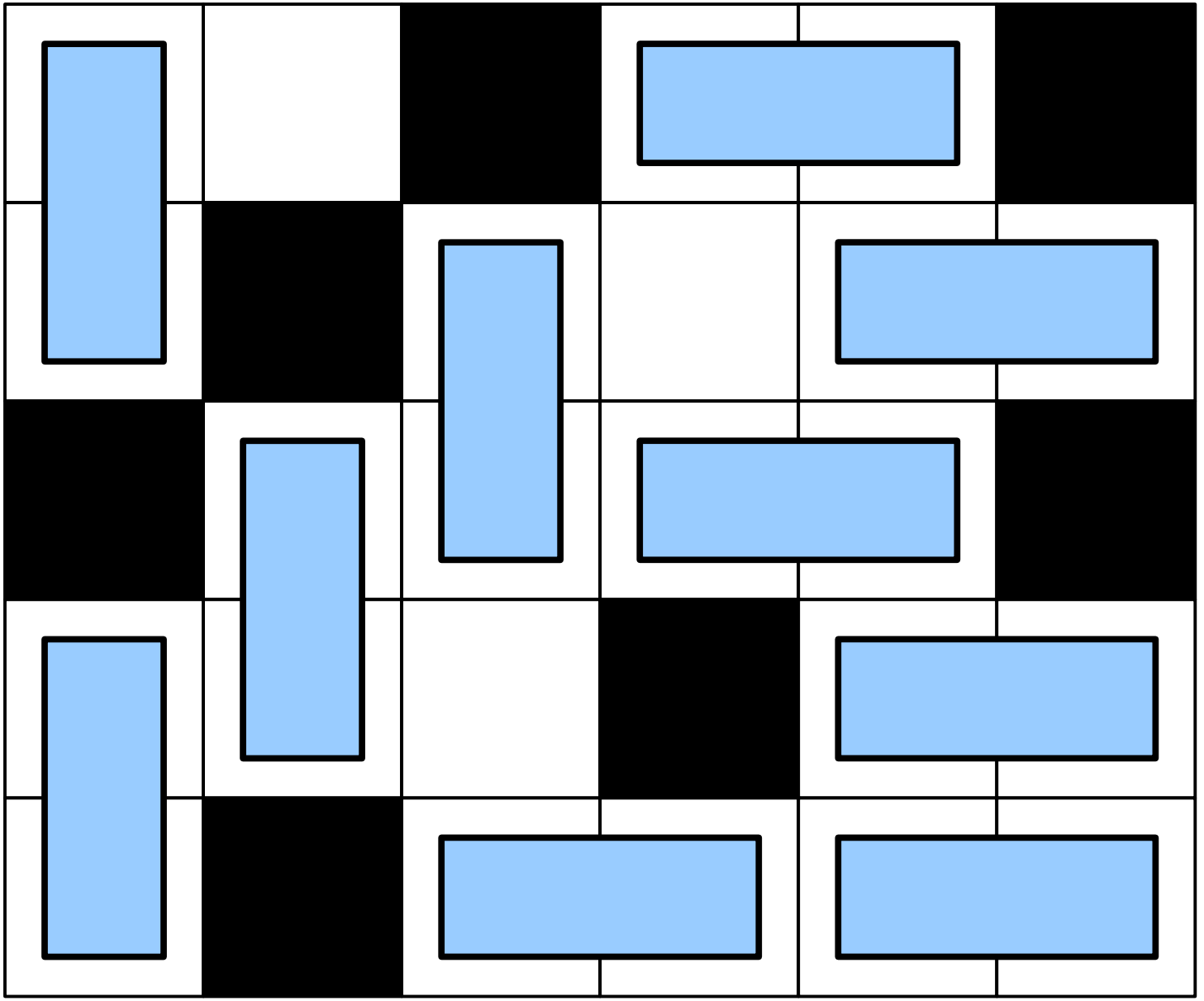

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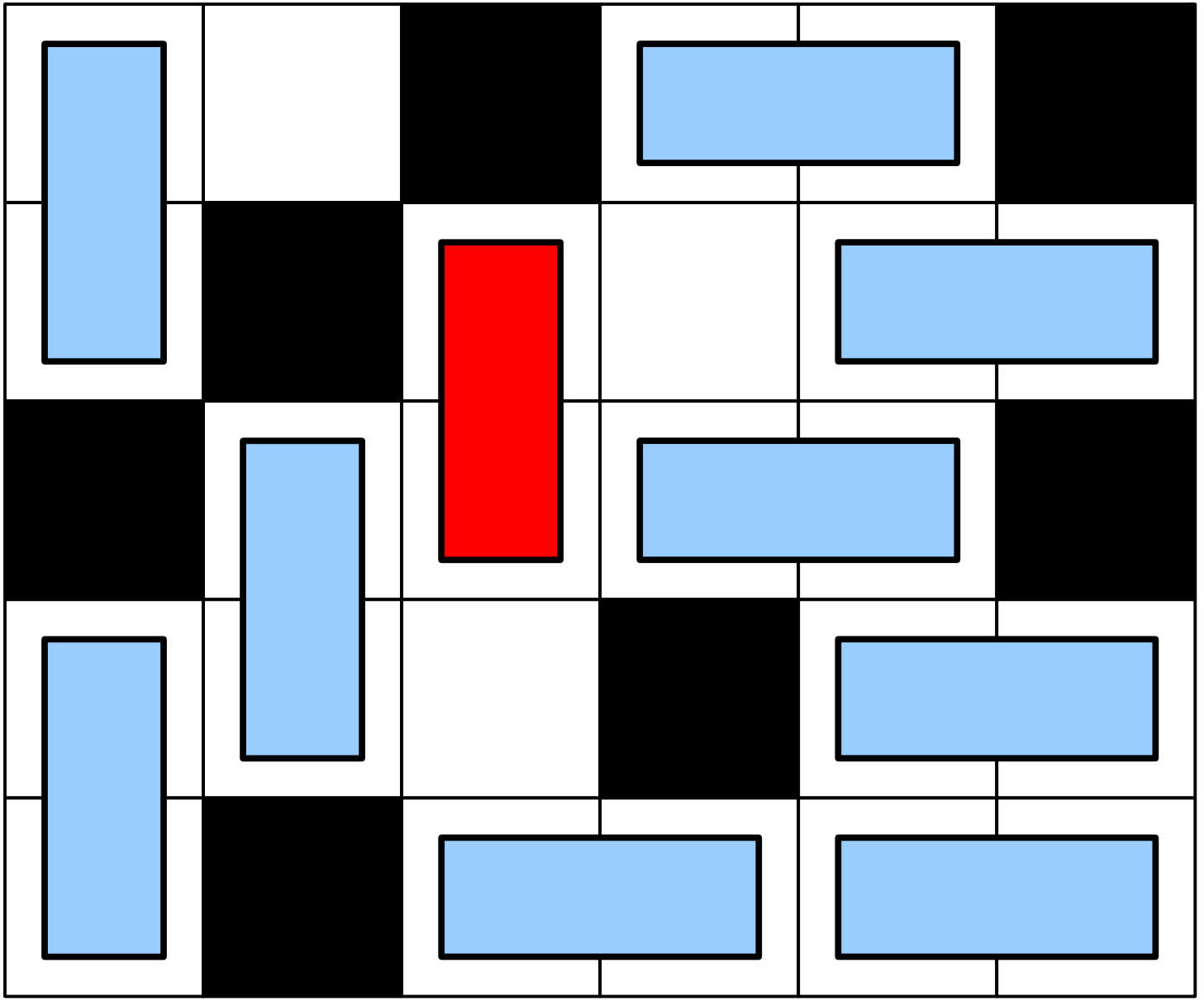

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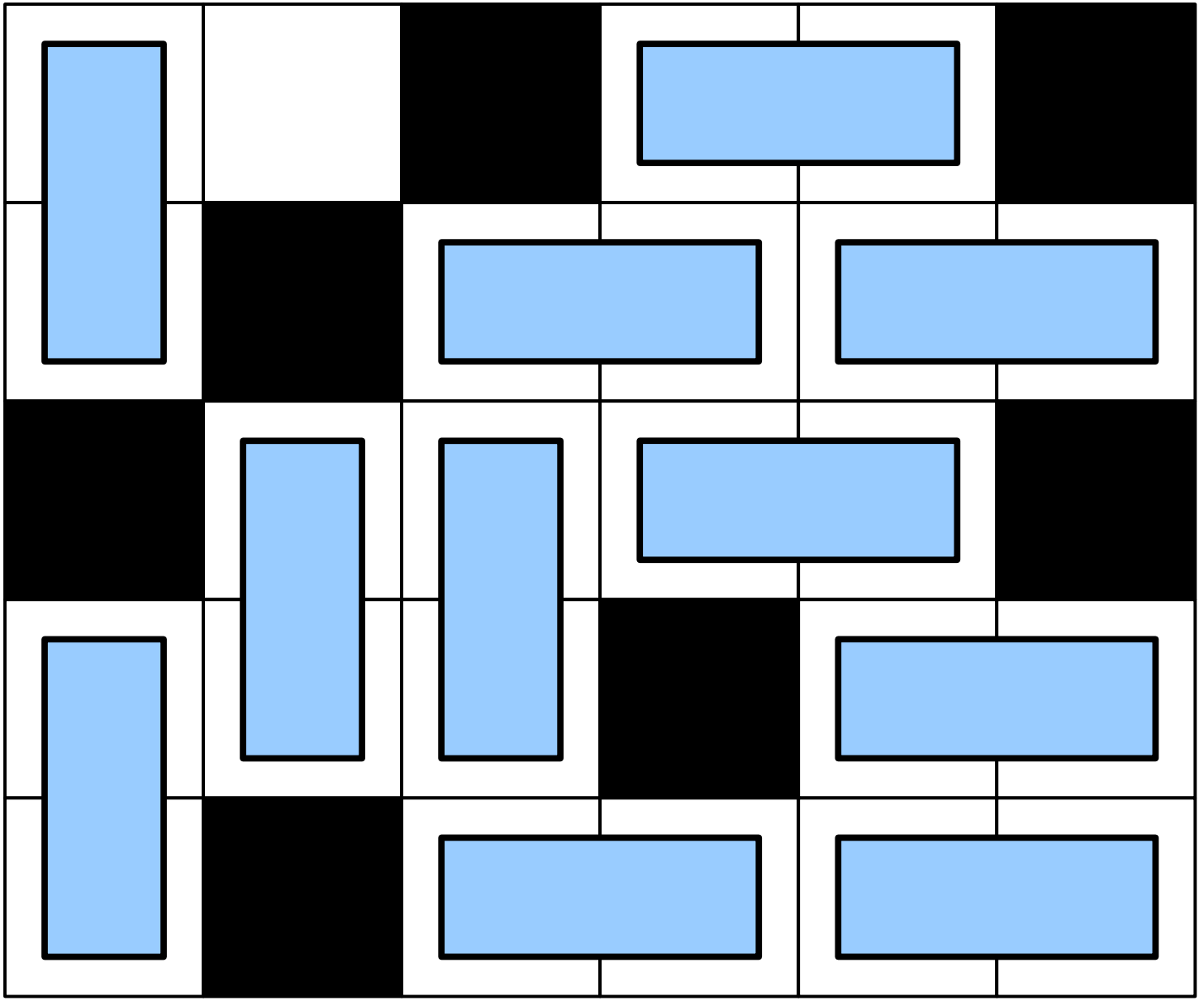
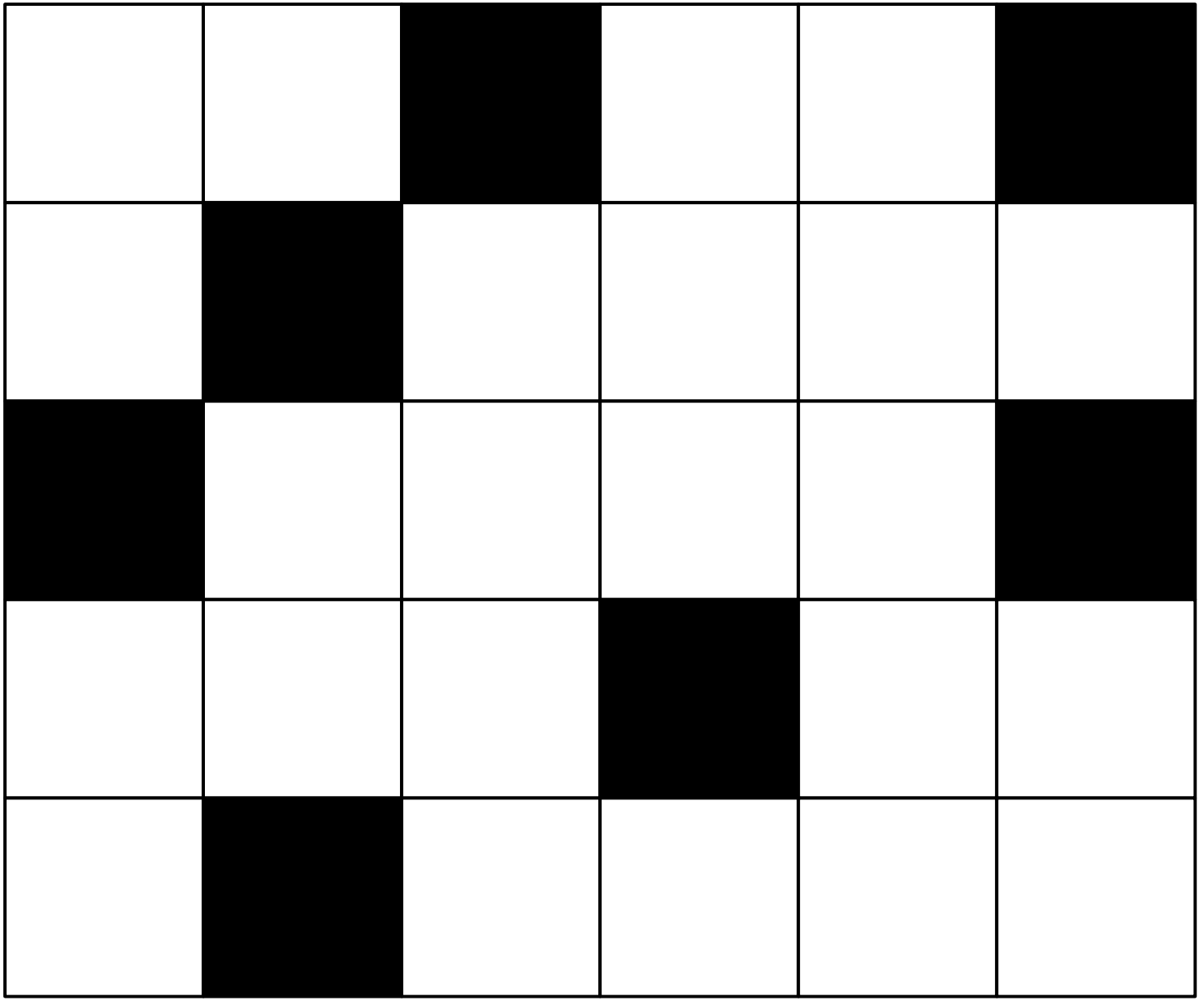


# Solving Domino Tiling



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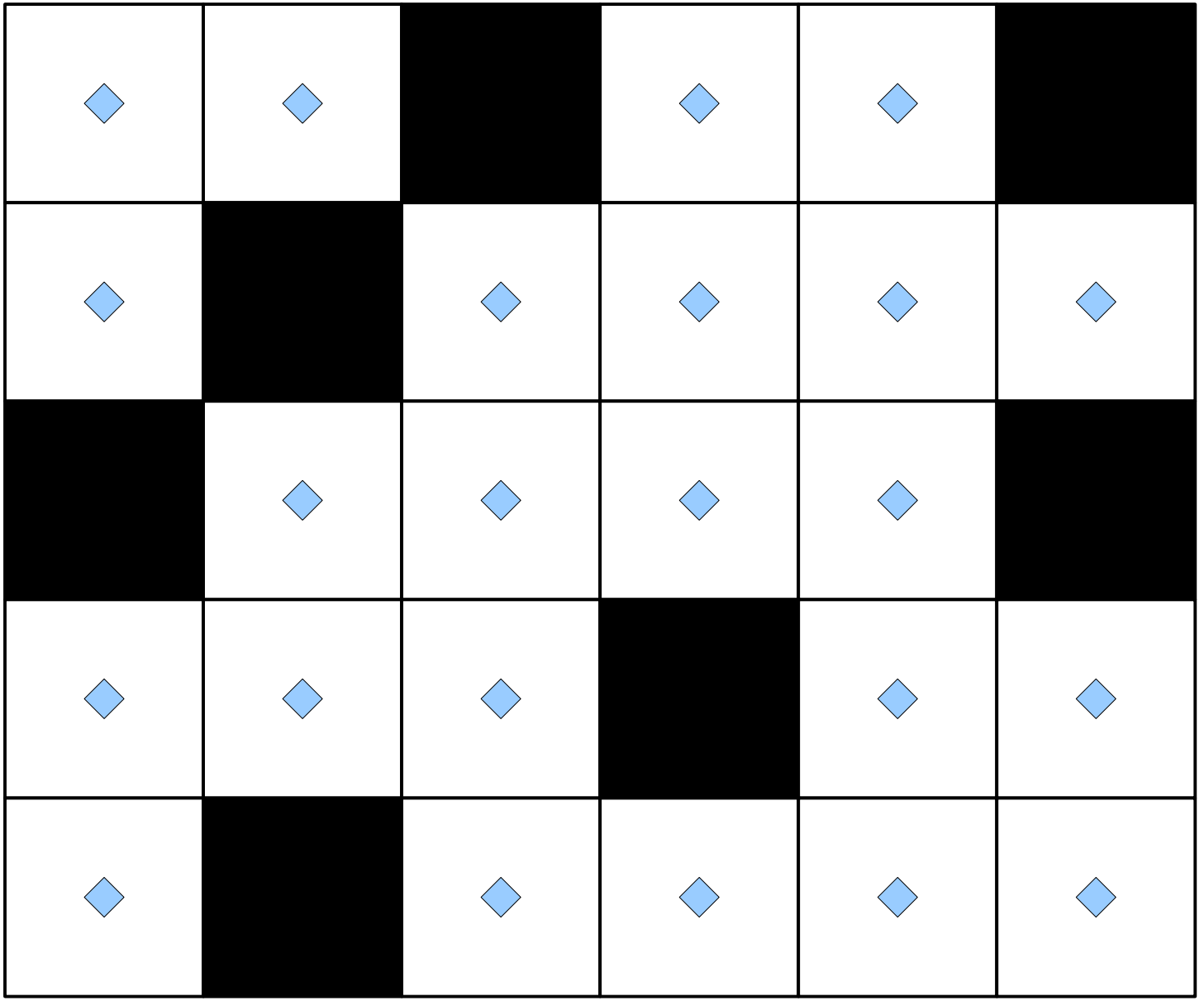

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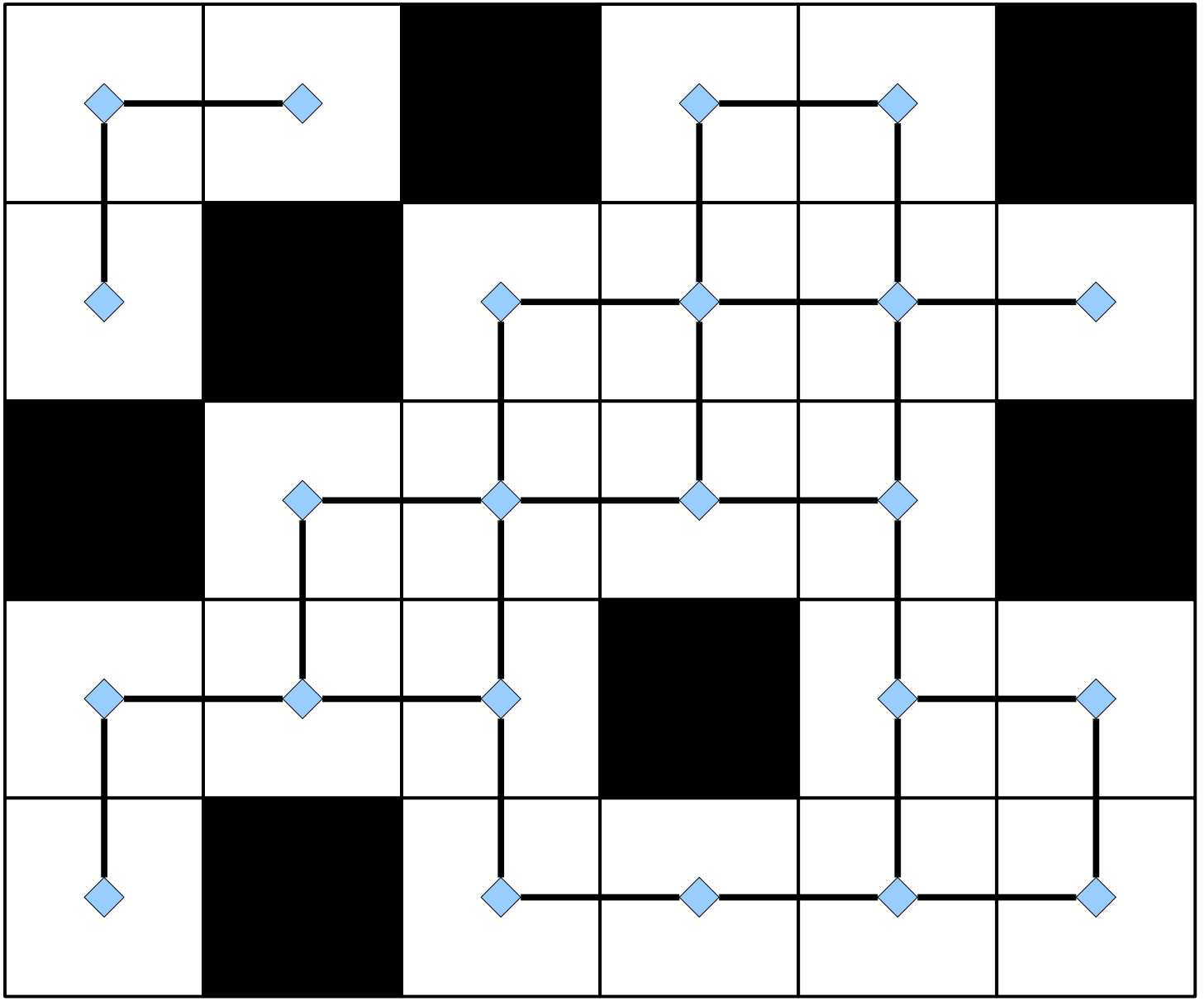

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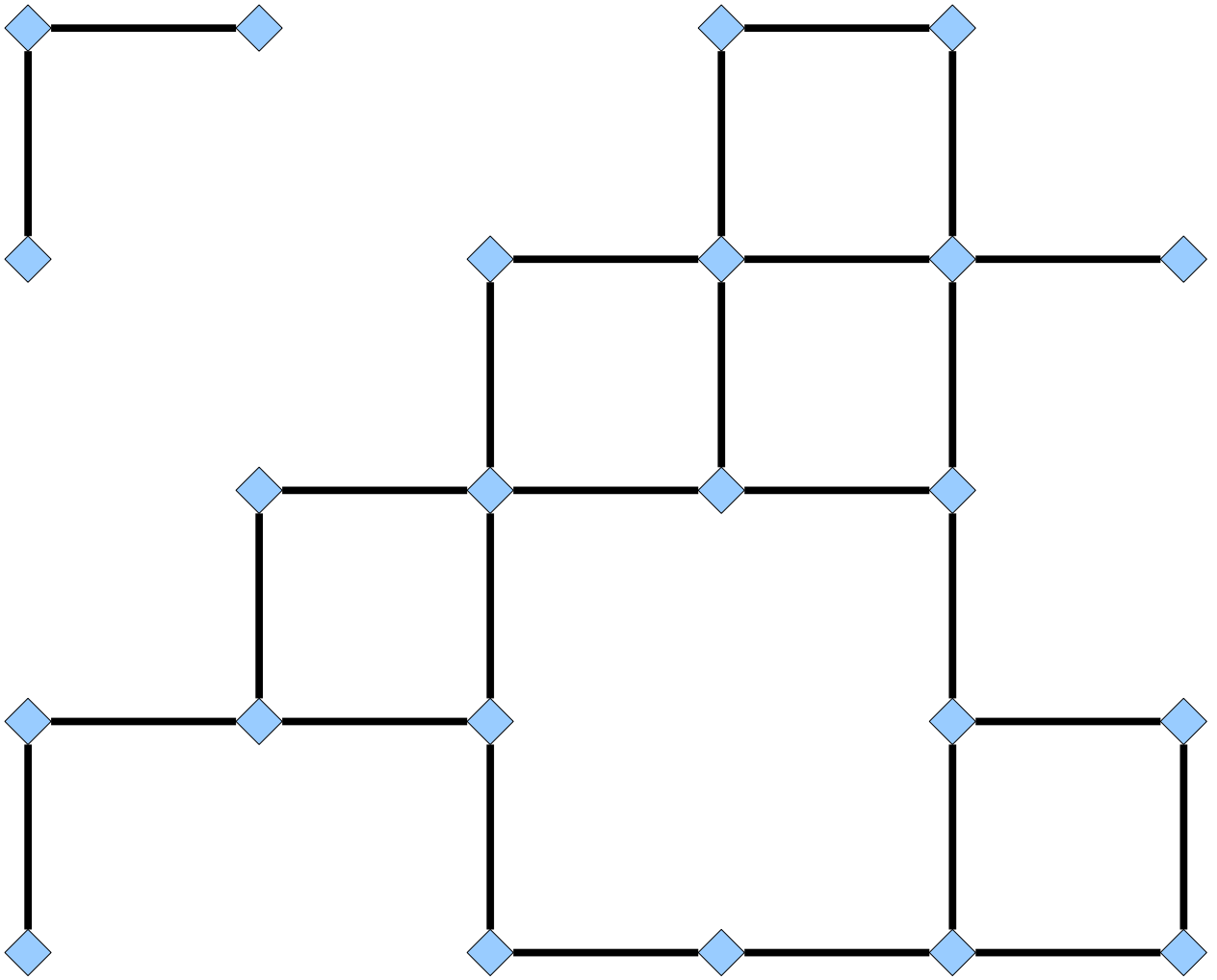

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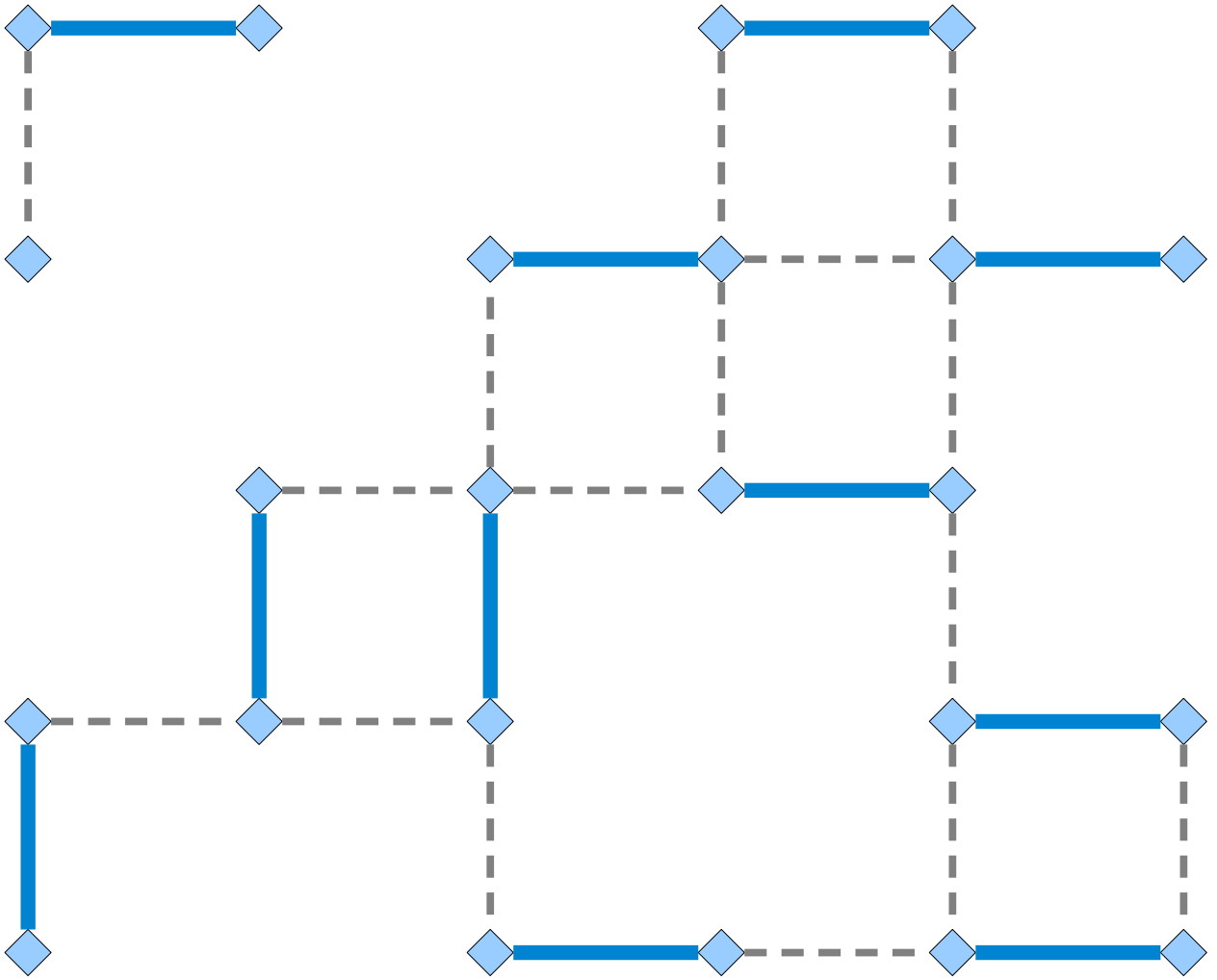

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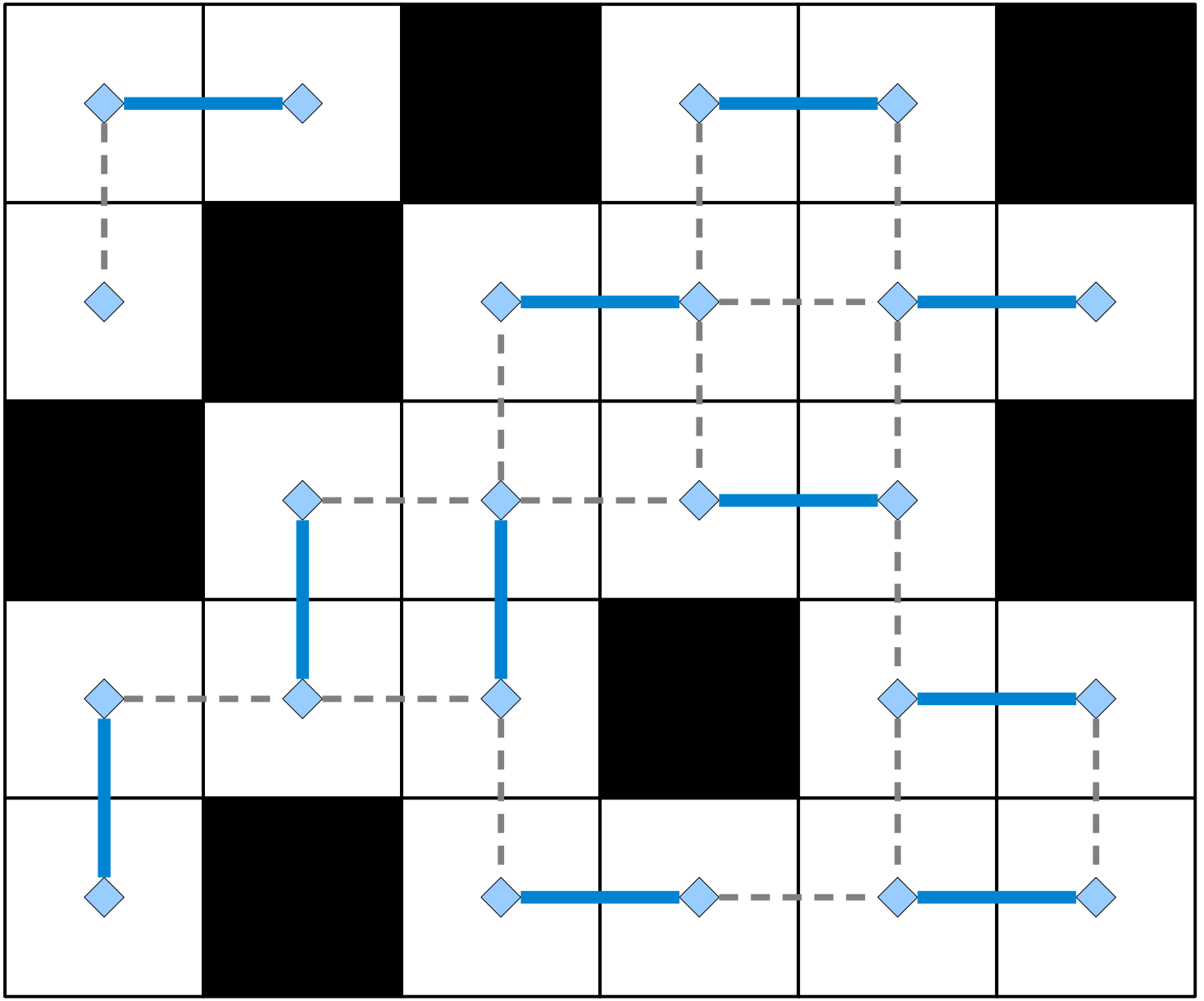

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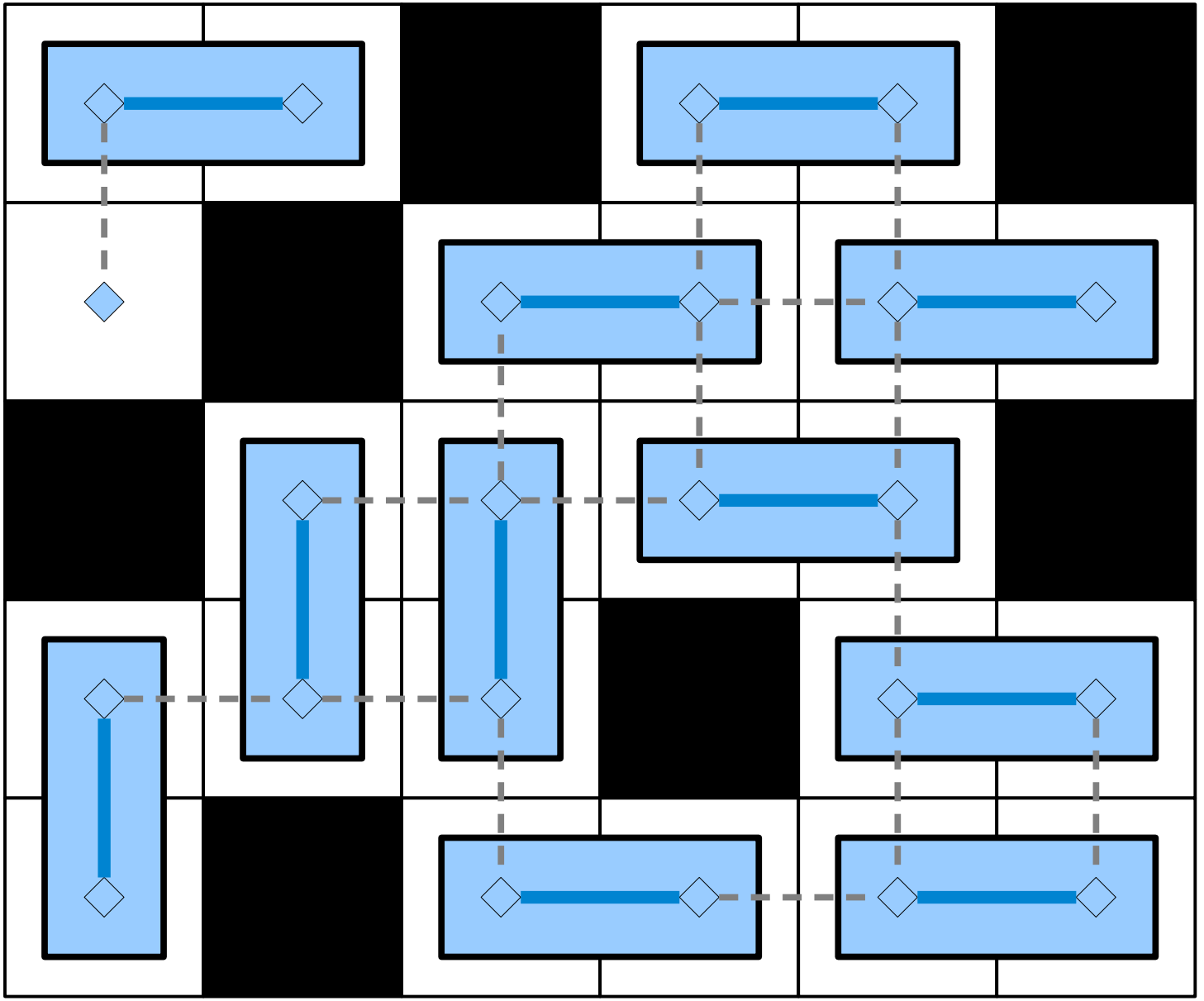

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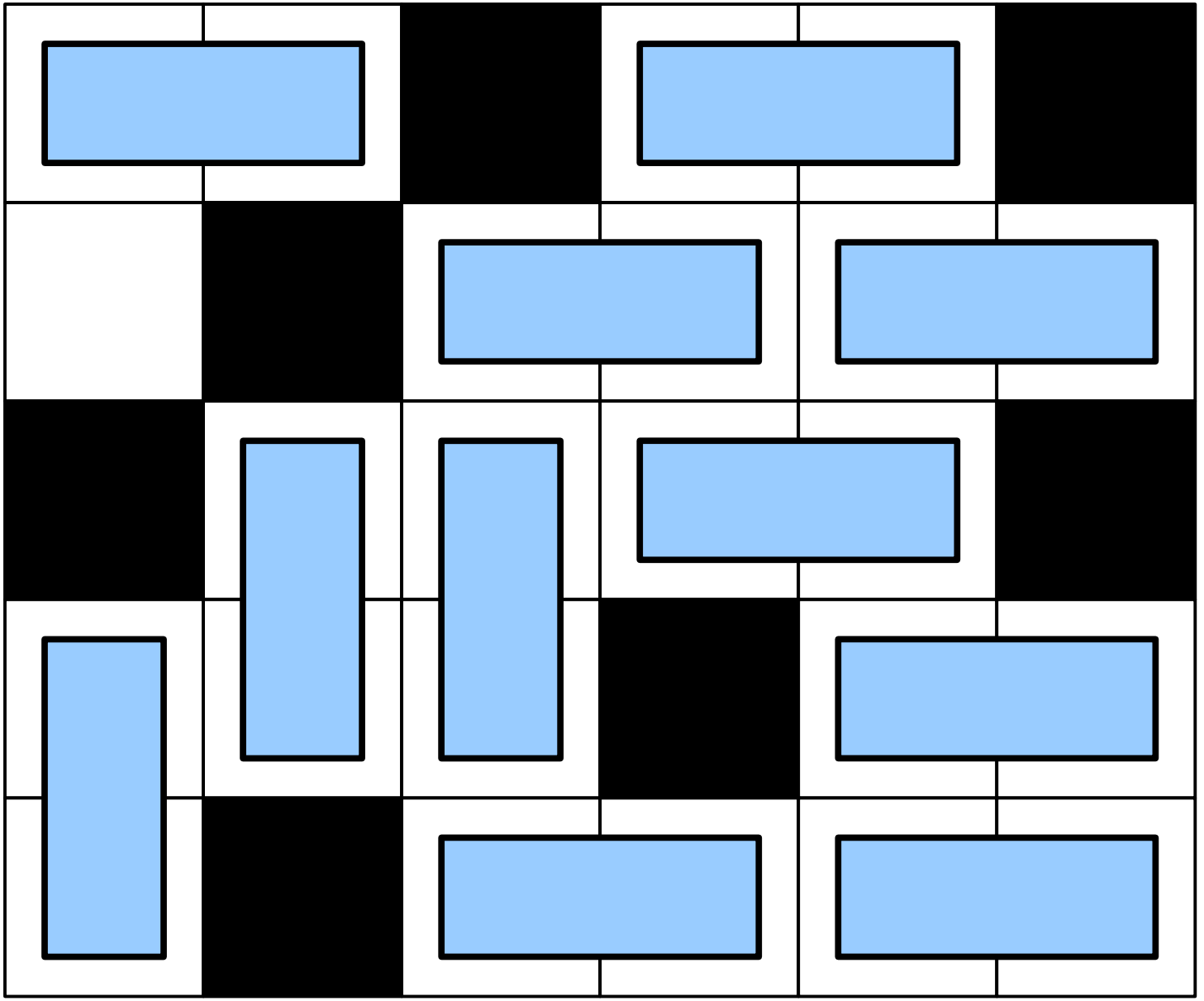


```
bool canPlaceDominoes(Grid G, int k) {
   return hasMatching(gridToGraph(G), k);
}
```

Which of the following is the most reasonable conclusion to draw, given the existence of the above function?

A. Solving domino tiling on a 2D grid can't be "harder" than solving maximum matching.

B. Solving maximum matching can't be "harder" than solving domino tiling on a 2D grid.

C. Both A and B.

Answer at
***https://cs103.stanford.edu/pollev***

## *Intuition:*

Tiling a grid with dominoes can't be “harder” than solving maximum matching, because if we can solve maximum matching efficiently, we can solve domino tiling efficiently.

# Another Example

# Satisfiability

- A propositional logic formula φ is called ***satisfiable*** if there is some assignment to its variables that makes it evaluate to true.
- Which of the following formulas are satisfiable?

$$p \wedge q$$

$$p \wedge \neg p$$

$$p \rightarrow (q \wedge \neg q)$$

- An assignment of true and false to the variables of φ that makes it evaluate to true is called a ***satisfying assignment***.

# SAT

- The ***boolean satisfiability problem*** (***SAT***) is the following:

**Given a propositional logic formula φ, is φ satisfiable?**

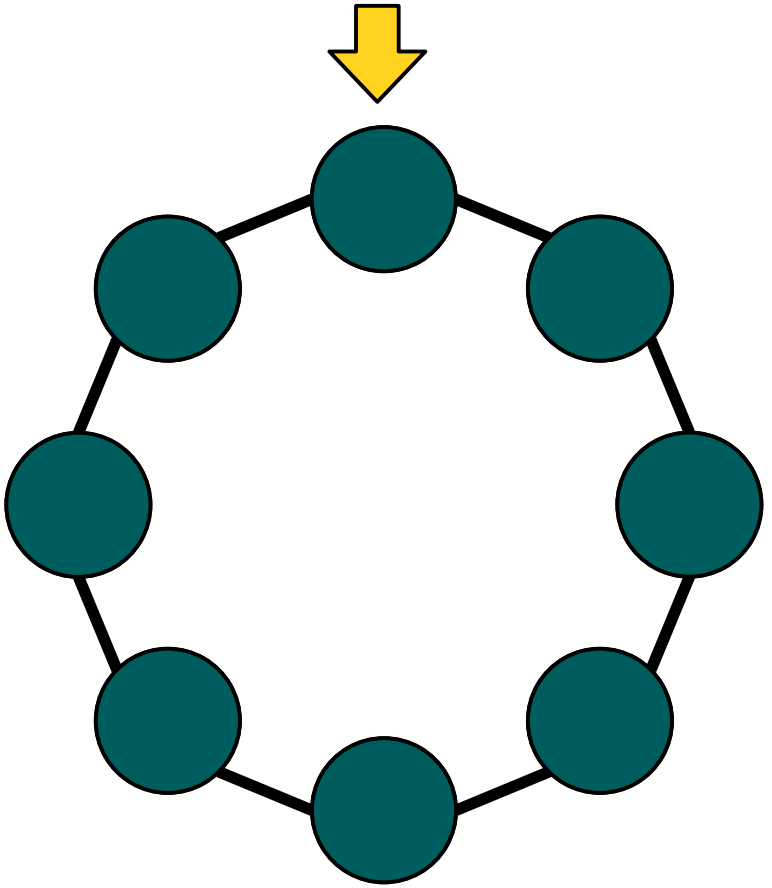
- Formally:

$$SAT = \{ \langle \varphi \rangle \mid \varphi \text{ is a satisfiable PL formula} \}$$

- Finding good algorithms for SAT is an active area of research for reasons we'll discuss later today.
- We have some pretty decent algorithms for solving SAT reasonably quickly most of the time.
- Given this, what other problems can we solve?

# Lights Out

- You’re given a ring of pushbuttons. Each pushbutton has a light that is either ON or OFF.
- If you push a button, it toggles the state of the two adjacent lights in the ring. (Lights that are ON turn OFF and vice-versa.)



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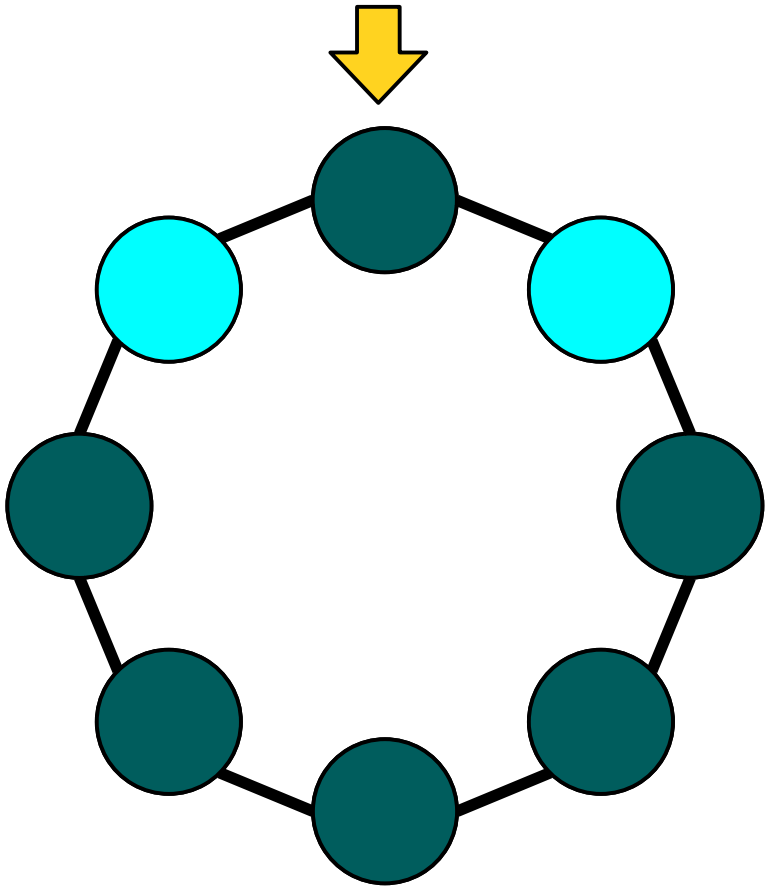

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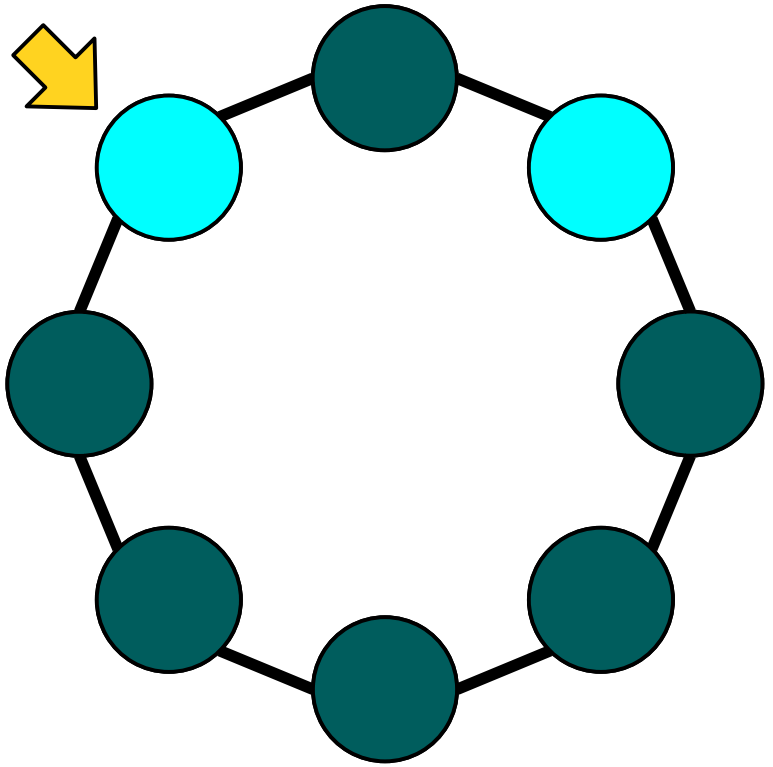

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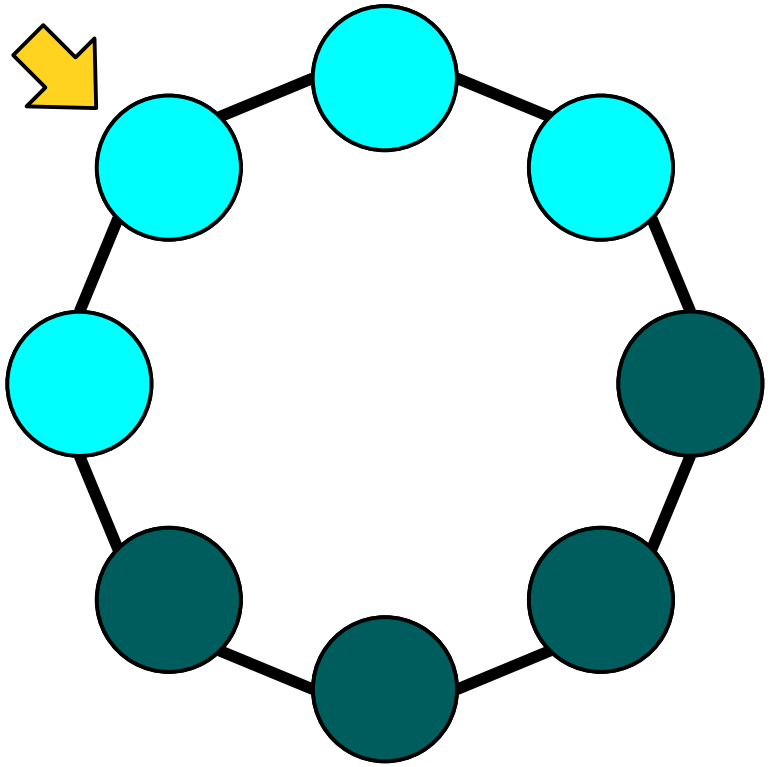

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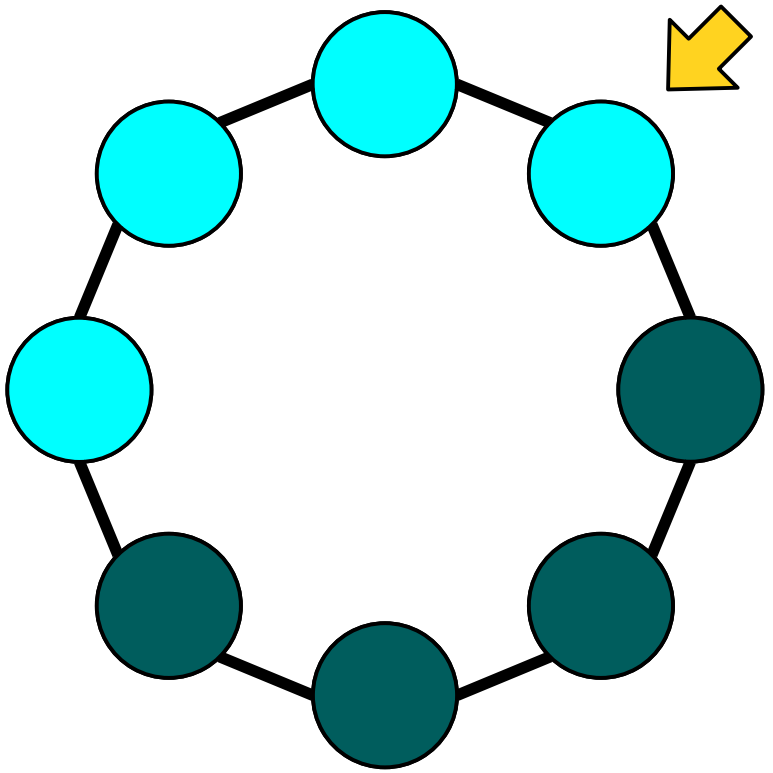

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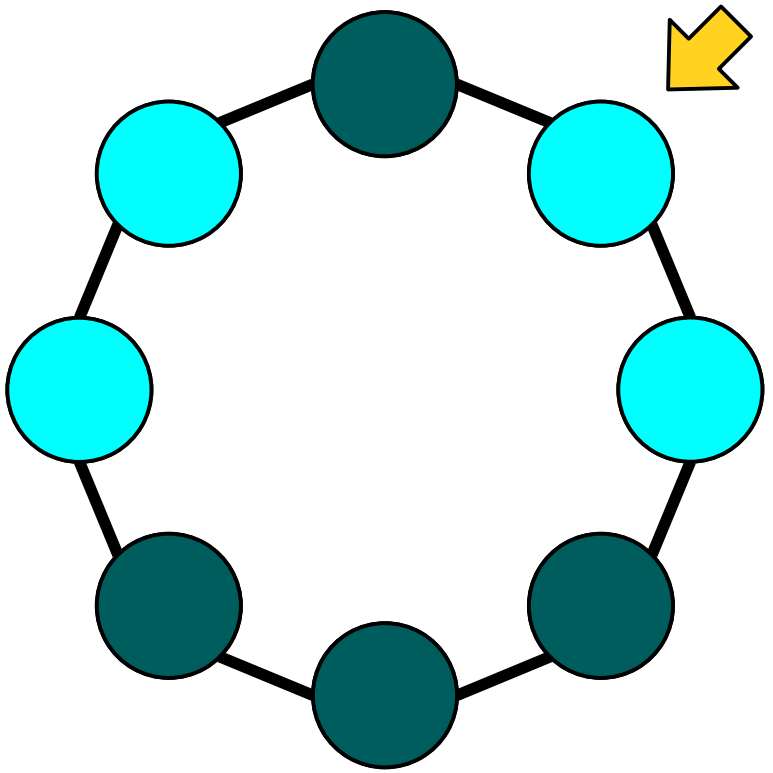

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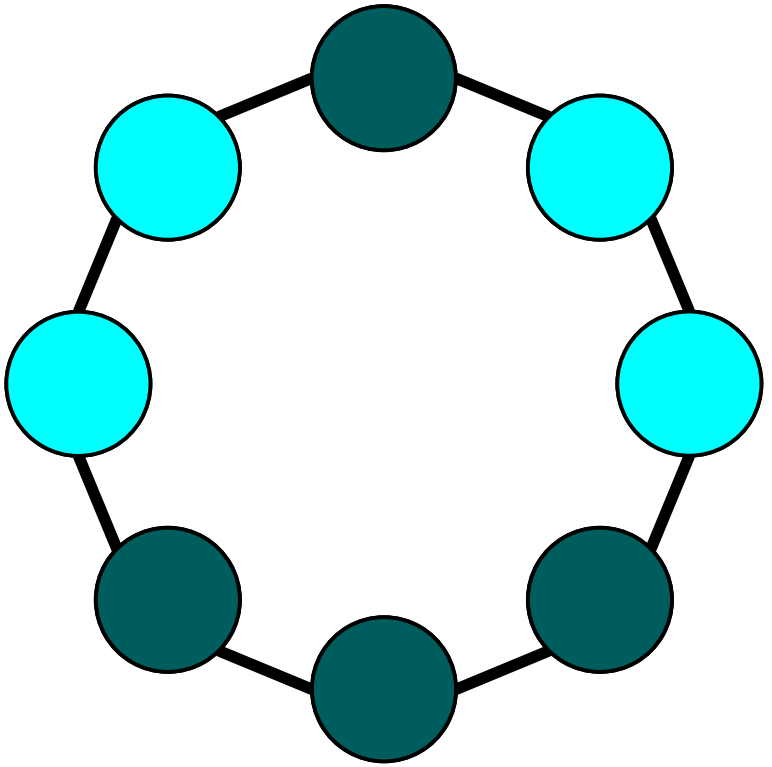
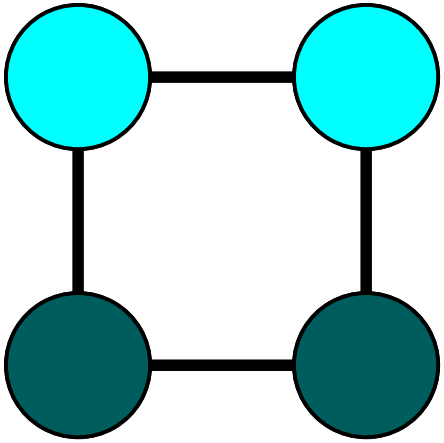
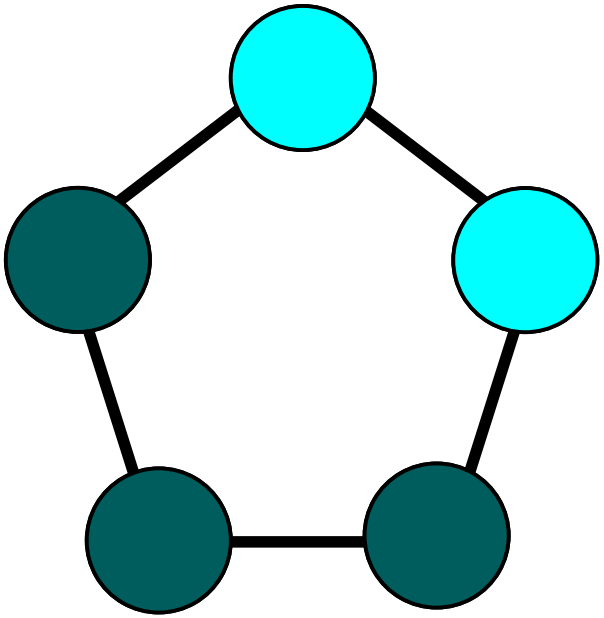
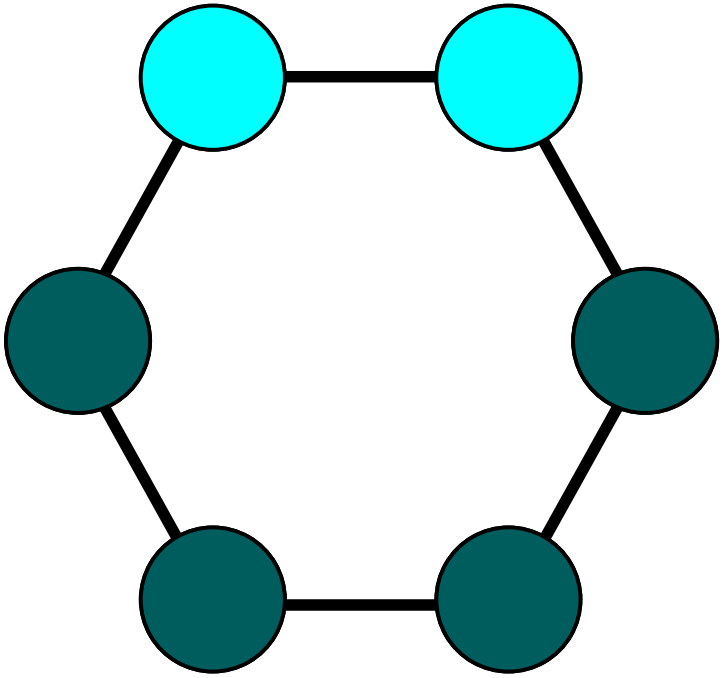
- You're given a ring of pushbuttons. Each pushbutton has a light that is either ON or OFF.
- If you push a button, it toggles the state of the two adjacent lights in the ring. (Lights that are ON turn OFF and vice-versa.)
- ***Question:*** Given an initial configuration of lights, can you turn all the lights off?



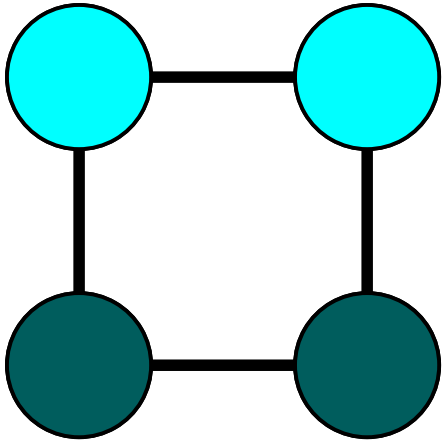


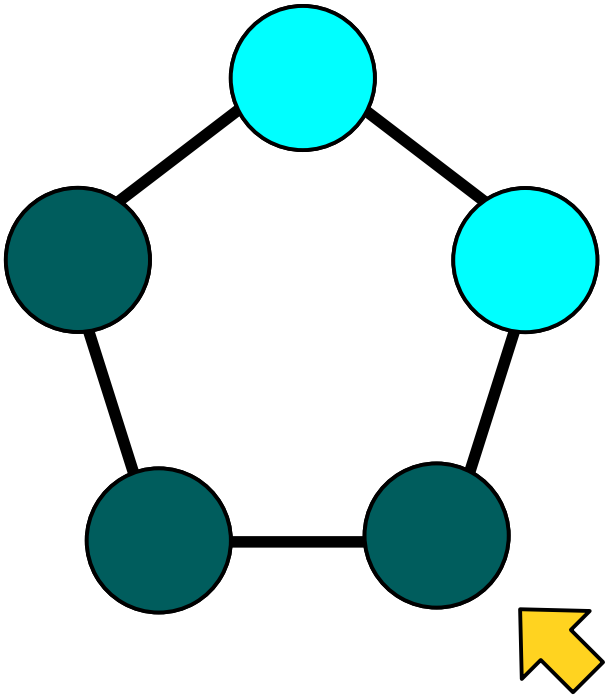


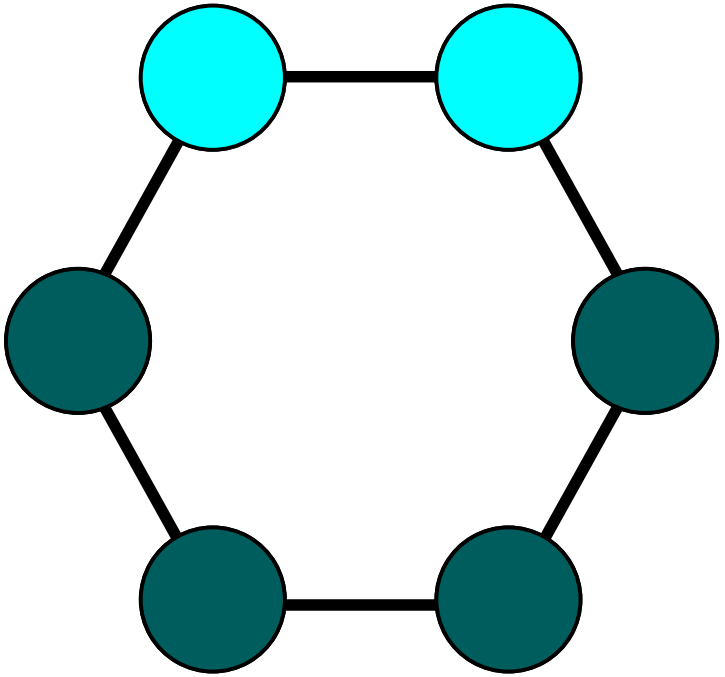


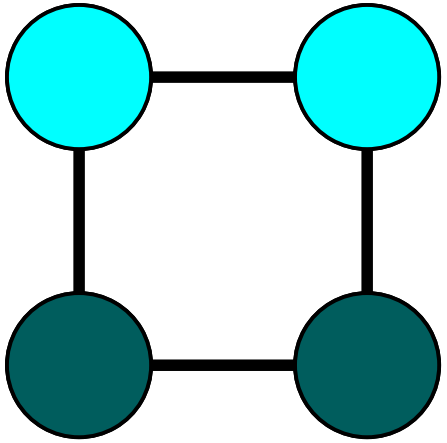
In which of these rings can you turn off all the lights?

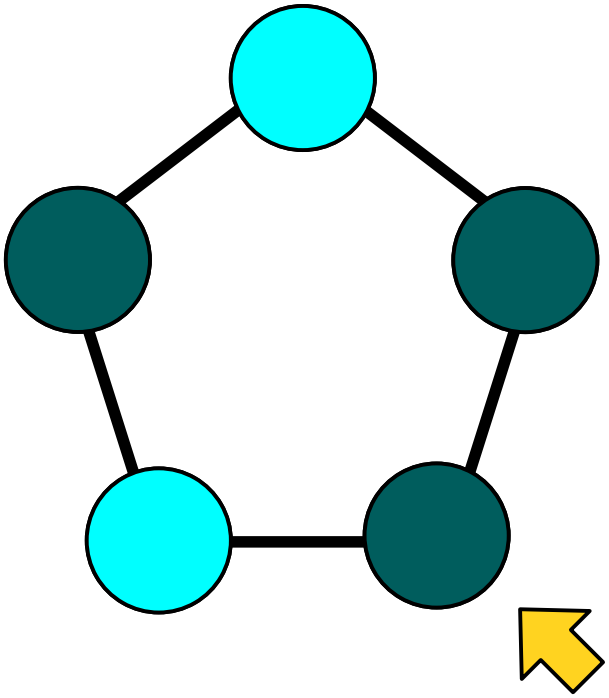
Answer at
***https://cs103.stanford.edu/pollev***

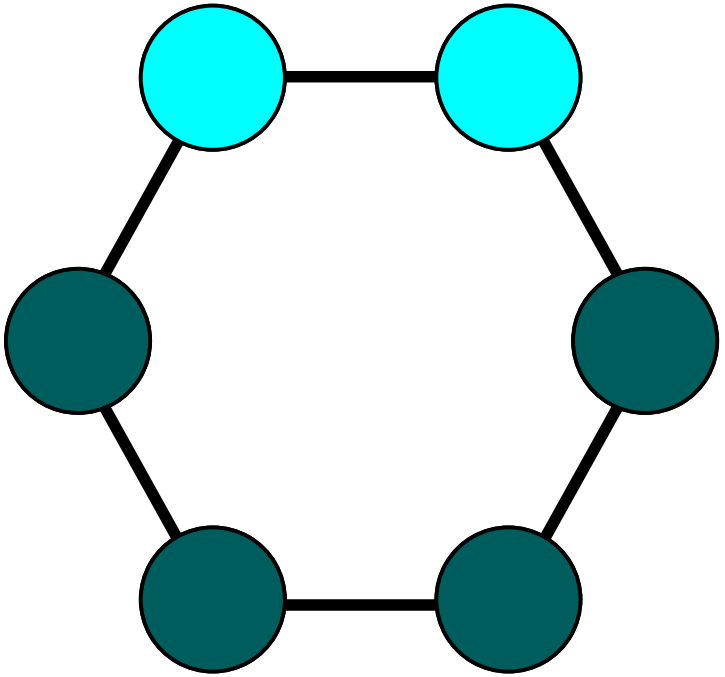


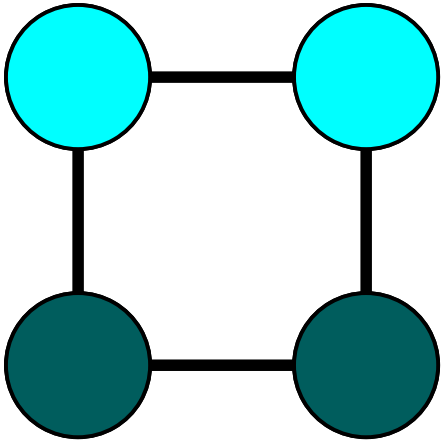


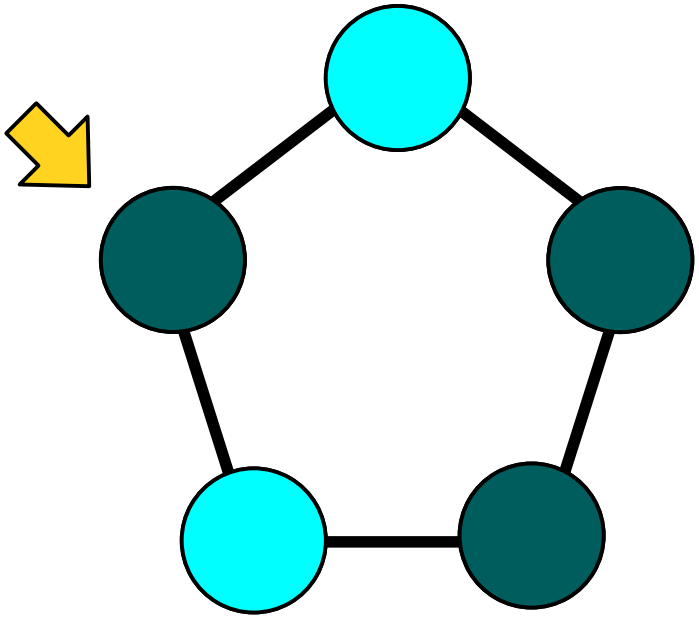


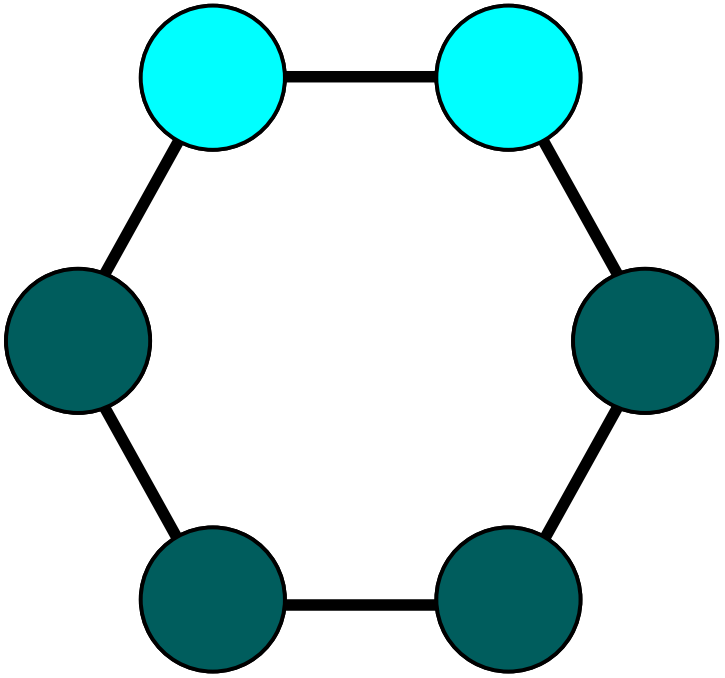


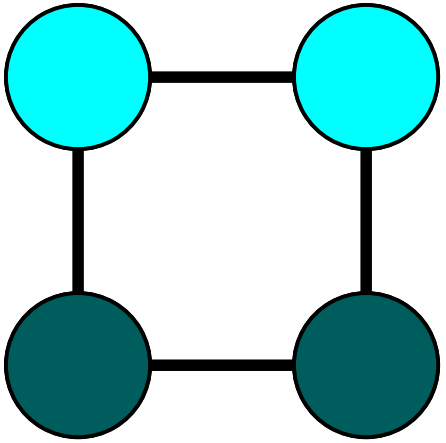


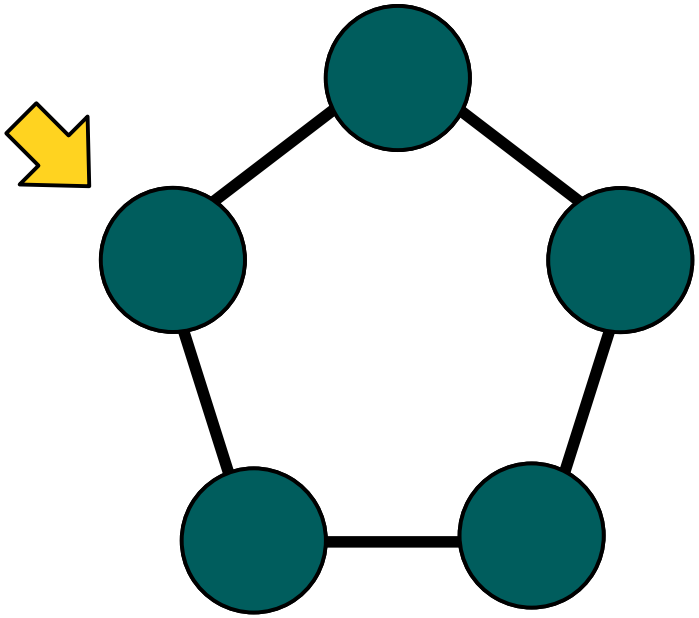


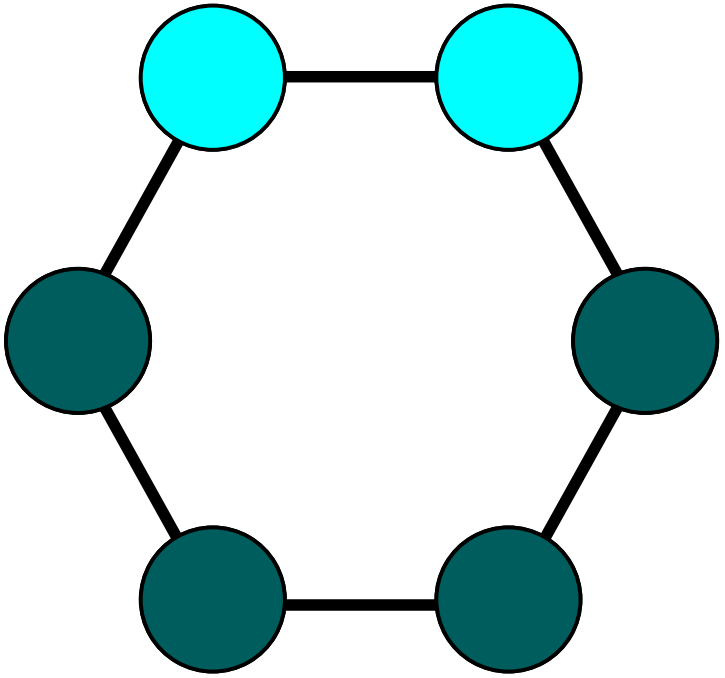


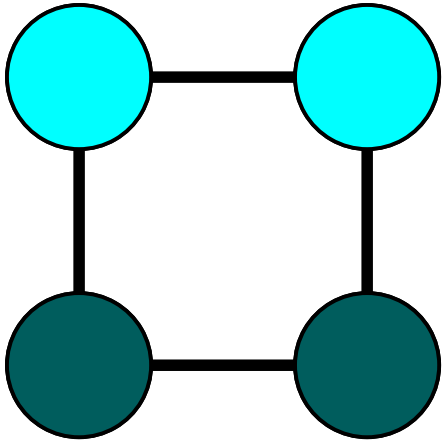


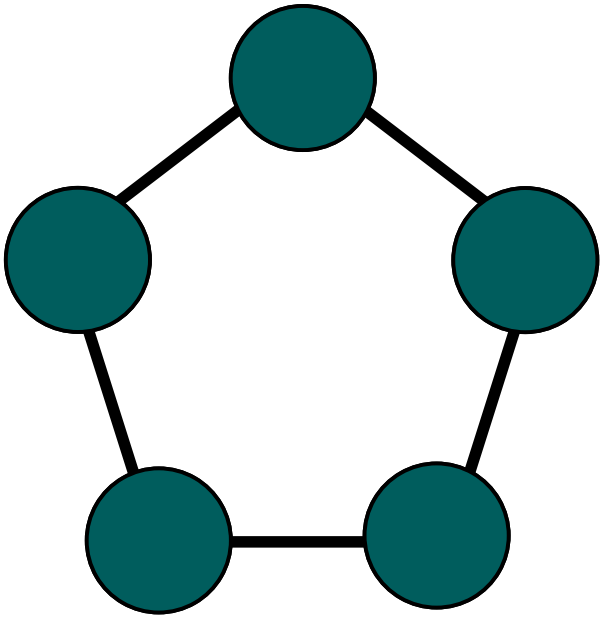


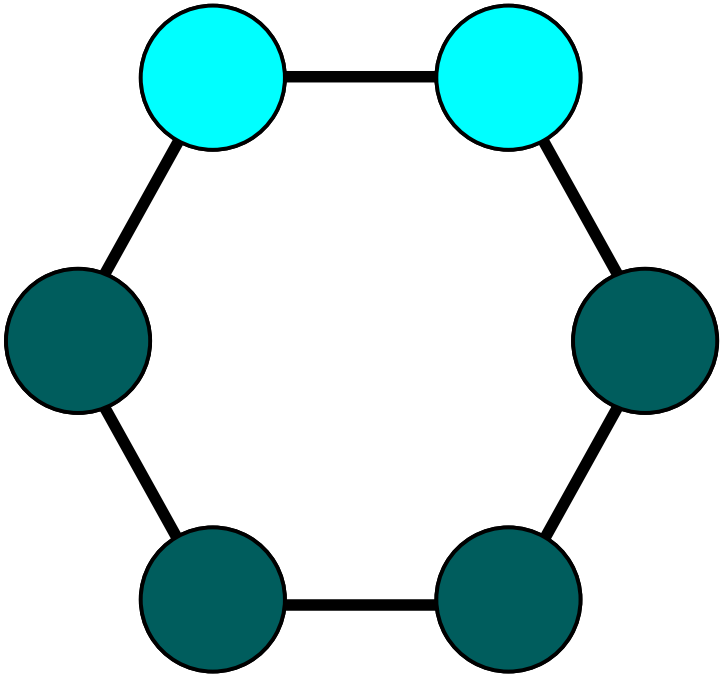




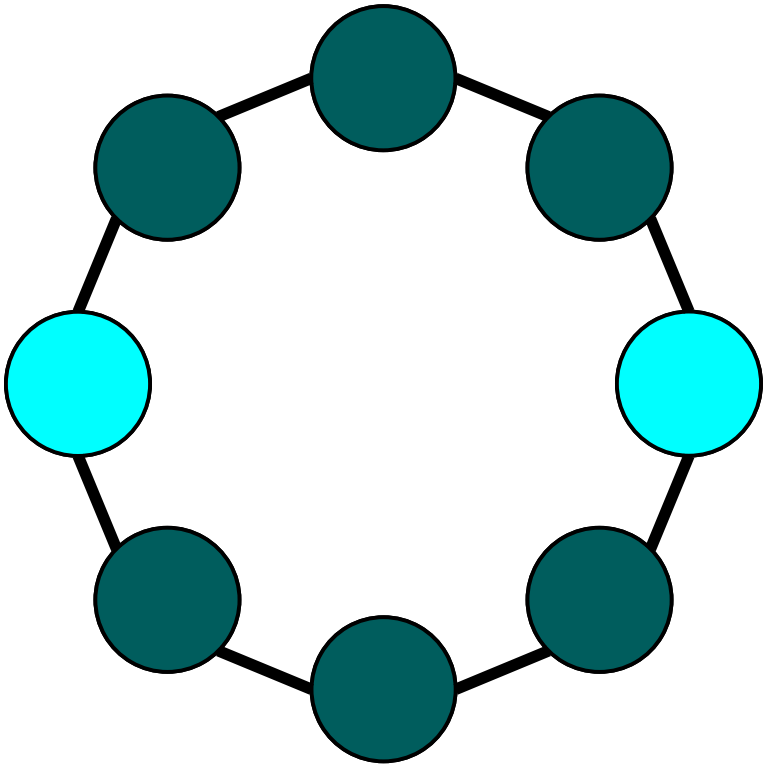


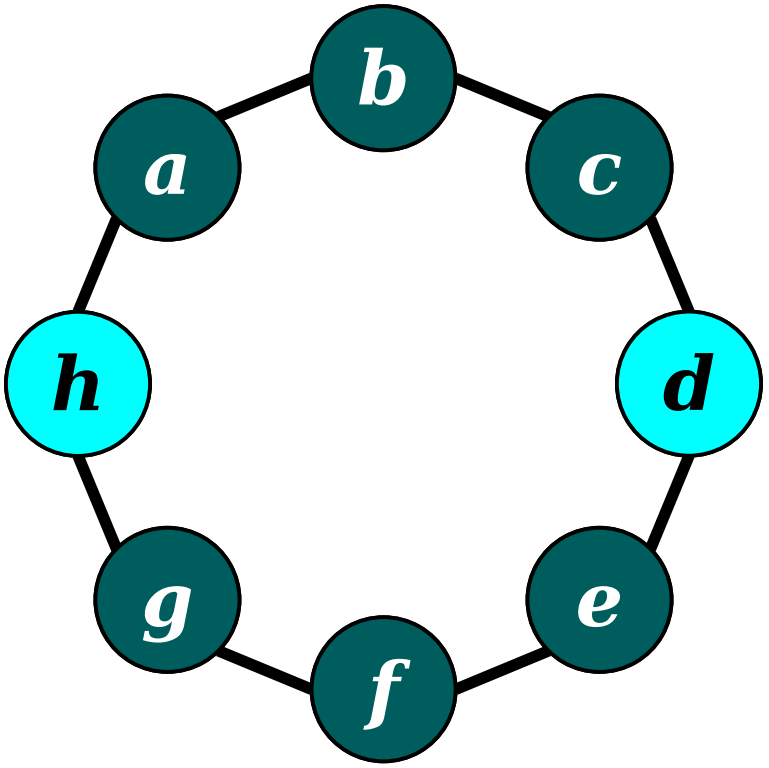






# Solving Lights-Out With a SAT Solver





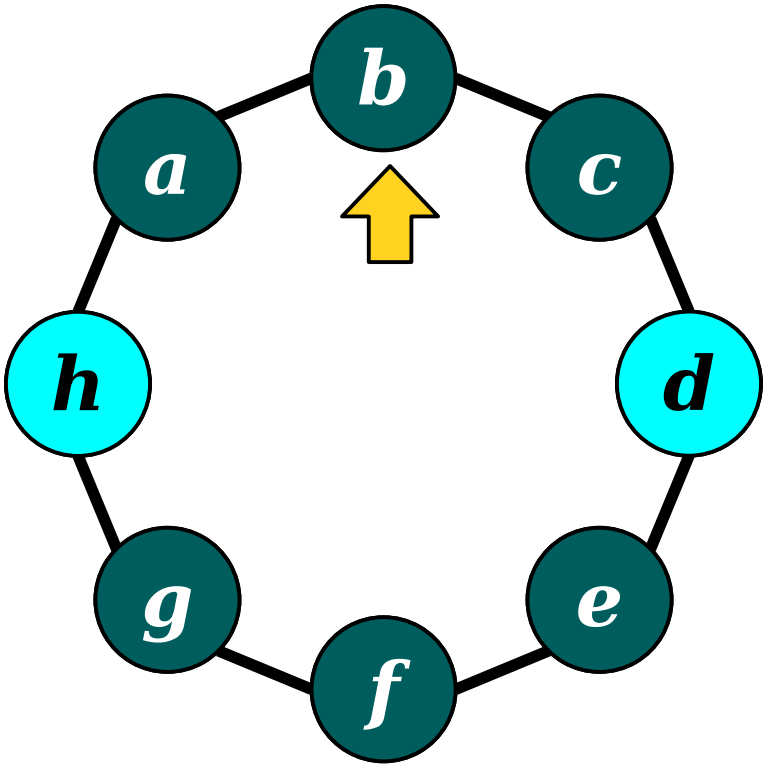

***Observation 1:*** We never need to press the same button twice.

***Observation 2:*** Button press order doesn't matter.

***Observation 3:*** Our propositional formula will have one variable per button, indicating whether we press it.

***Observation 4:*** A light that is initially off stays off when an even number of adjacent lights are pressed.

***Observation 5:*** A light that is initially on ends off when an odd number of adjacent lights are pressed.




**Observation 1:** We never need to press the same button twice.

**Observation 2:** Button press order doesn't matter.

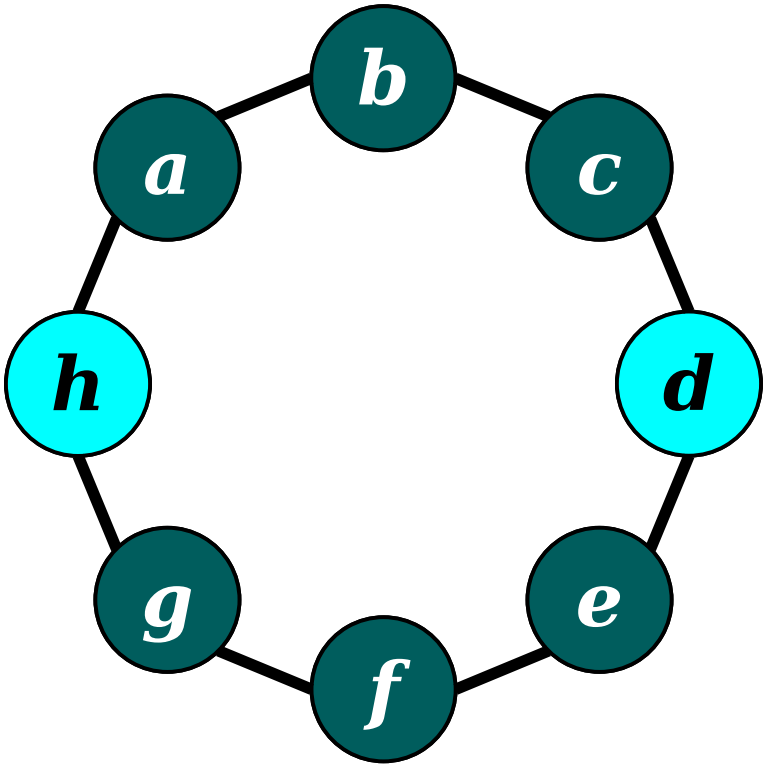
**Observation 3:** Our propositional formula will have one variable per button, indicating whether we press it.

**Observation 4:** A light that is initially off stays off when an even number of adjacent lights are pressed.

**Observation 5:** A light that is initially on ends off when an odd number of adjacent lights are pressed.

Write a formula in propositional logic that says "an even number of the variables *a* and *c* are true."

Answer at
https://cs103.stanford.edu/pollev




$$
\begin{aligned}
(h \leftrightarrow b) \ &\wedge \\
(a \leftrightarrow c) \ &\wedge \\
(b \leftrightarrow d) \ &\wedge \\
\neg(c \leftrightarrow e) \ &\wedge \\
(d \leftrightarrow f) \ &\wedge \\
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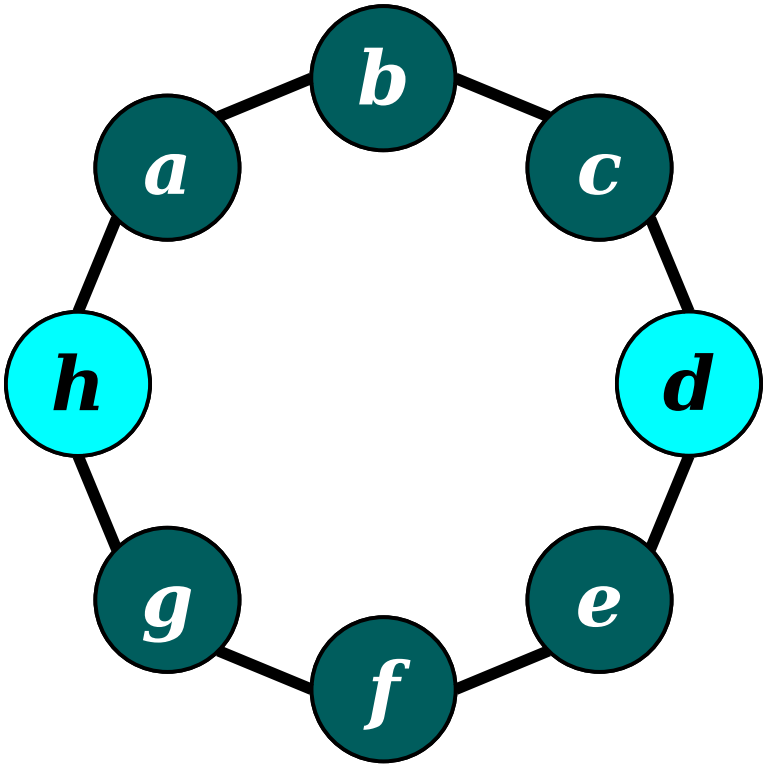


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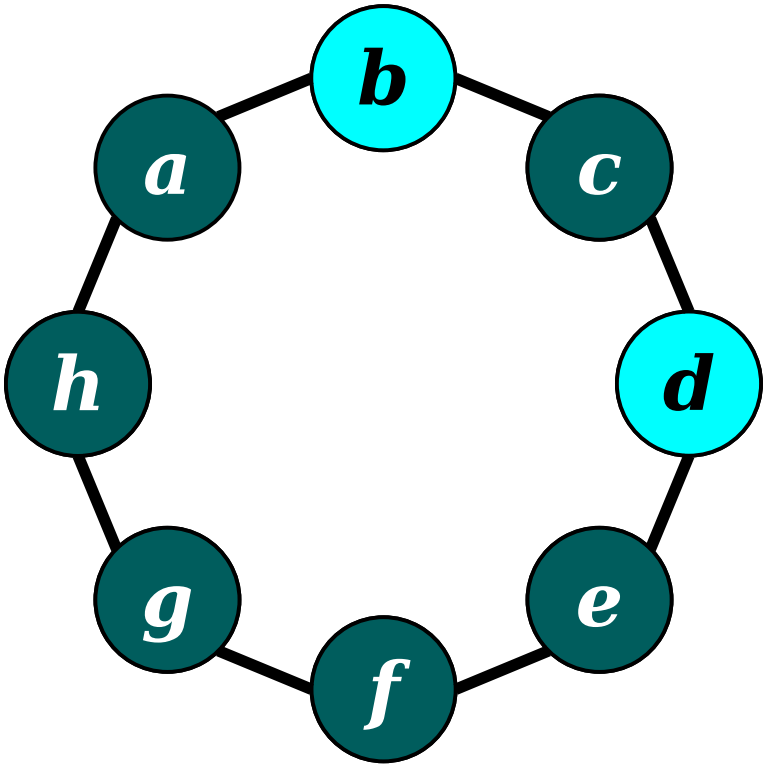


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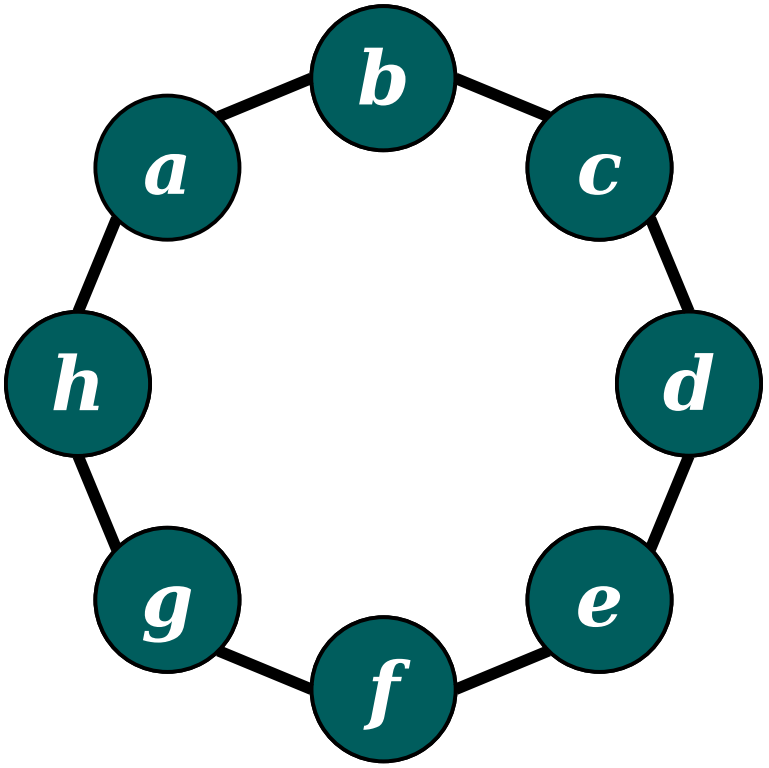


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# In Pseudocode

```
bool canTurnLightsOff(LightRing r) {
  return isSatisfiable(ringToFormula(r));
}
```

***Intuition:***

Solving Lights Out can’t be “harder” than solving SAT because if we can solve SAT efficiently, we can solve Lights Out efficiently.

```
bool canPlaceDominoes(Grid G, int k) {
   return hasMatching(gridToGraph(G), k);
}

bool canTurnLightsOff(LightRing r) {
   return isSatisfiable(ringToFormula(r));
}
```

```
bool solveProblemA(string input) {
    return solveProblemB(translate(input));
}
```

***Intuition:***

Problem *A* can't be "harder" than problem *B*, because solving problem *B* lets us solve problem *A*.

```
bool solveProblemA(string input) {
    return solveProblemB(translate(input));
}
```

- If $A$ and $B$ are problems where it's possible to solve problem $A$ using the strategy shown above*, we write

$$A \leq_{\mathrm{p}} B.$$

- We say that ***A is polynomial-time reducible to B***.

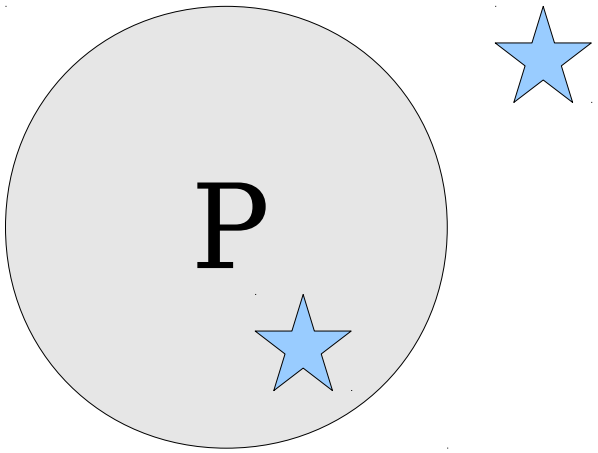
* Assuming that `translate` runs in polynomial time.

```
bool solveProblemA(string input) {
    return solveProblemB(translate(input));
}
```

- This is a powerful general problem-solving technique. You'll see it a lot in CS161.

# Polynomial-Time Reductions

- If $A \leq_p B$ and $B \in \mathbf{P}$, then $A \in \mathbf{P}$.



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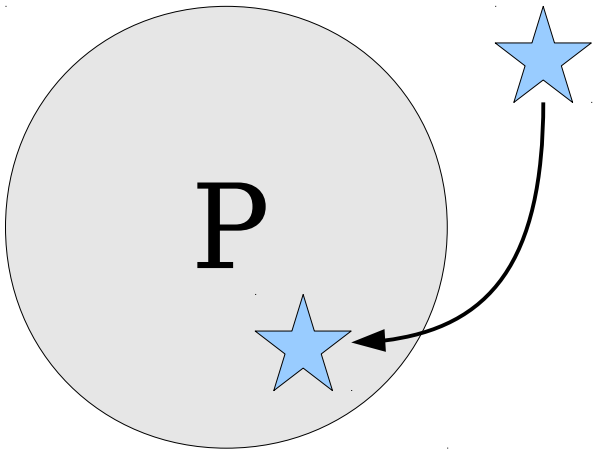

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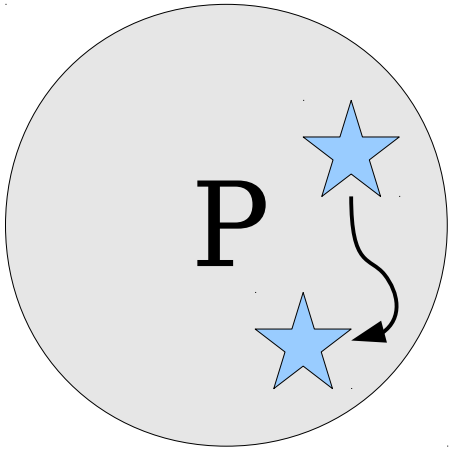
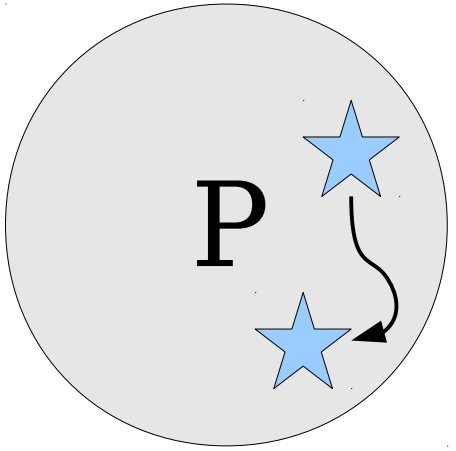


# Polynomial-Time Reductions

- If $A \leq_p B$ and $B \in \mathbf{P}$, then $A \in \mathbf{P}$.
- If $A \leq_p B$ and $B \in \mathbf{NP}$, then $A \in \mathbf{NP}$.



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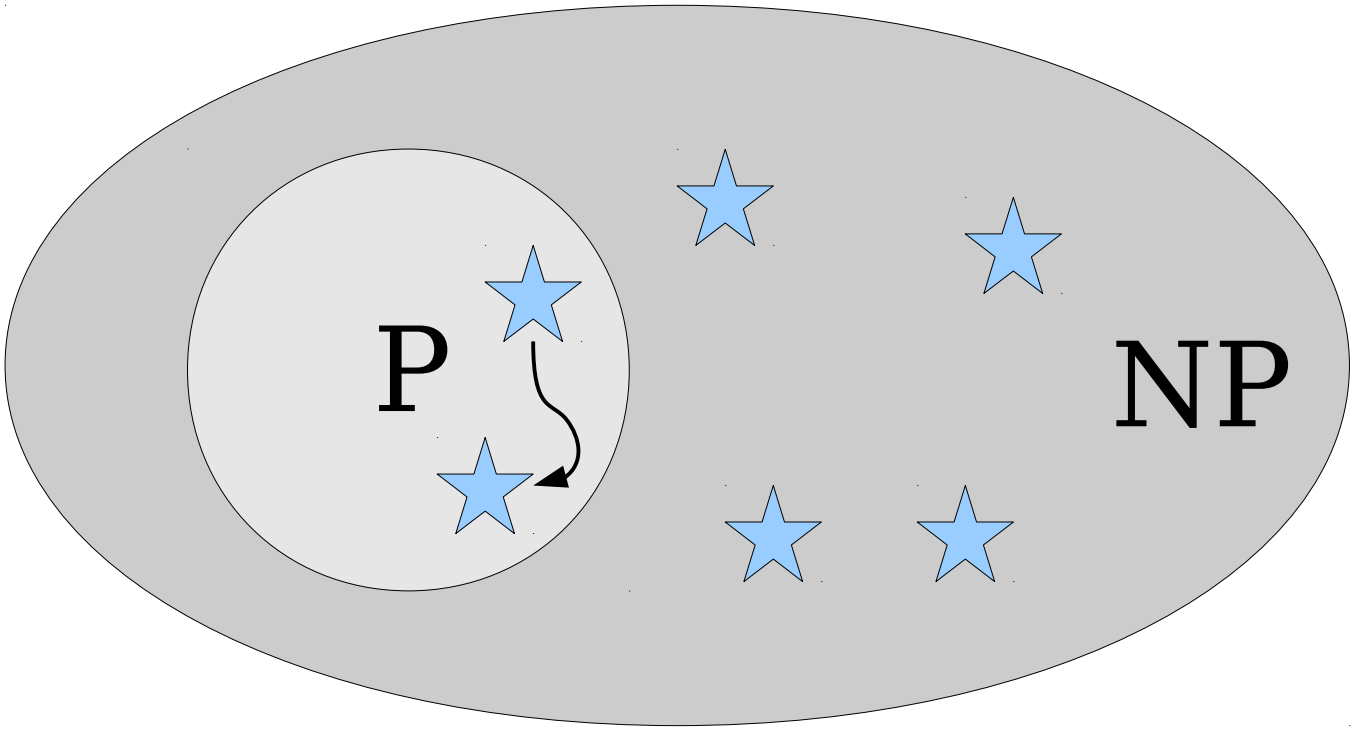

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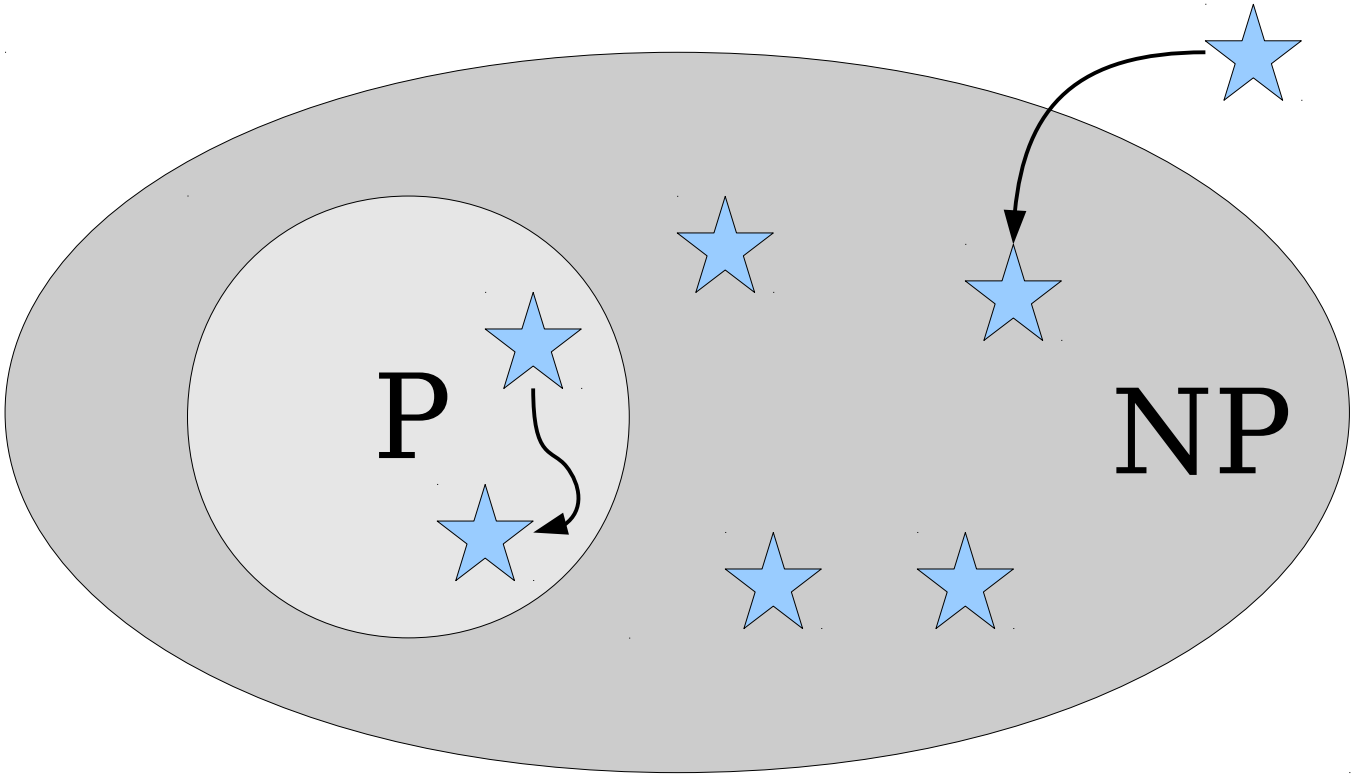

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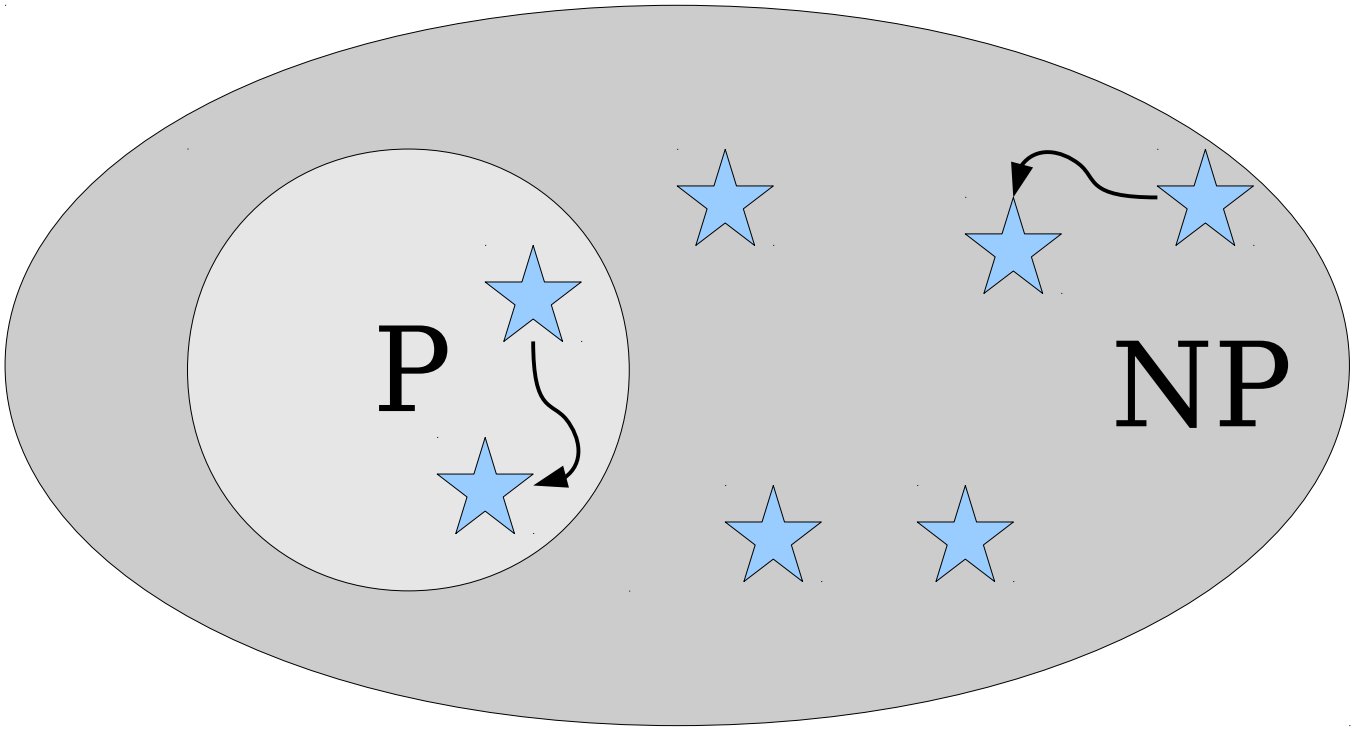


This $\leq_p$ relation lets us rank the relative difficulties of problems in **P** and **NP**.

What else can we do with it?

Time-Out for Announcements!

***Please evaluate this course on Axess.***

Your feedback makes a difference.

# Final Exam Logistics

- Our final exam is on ***Wednesday, March 19th*** from ***3:30 – 6:30 PM***.
  - Seating assignments will be online soon; we'll make an announcement when they're ready.
- The final exam is cumulative, covering topics from PS0 – PS9 and L00 – L26. The format is similar to that of the midterms, with a mix of short-answer questions and formal written proofs.
- Like the midterms, it's closed-book, closed-computer, and limited-note. You can bring one double-sided 8.5" × 11" notes sheet with you.

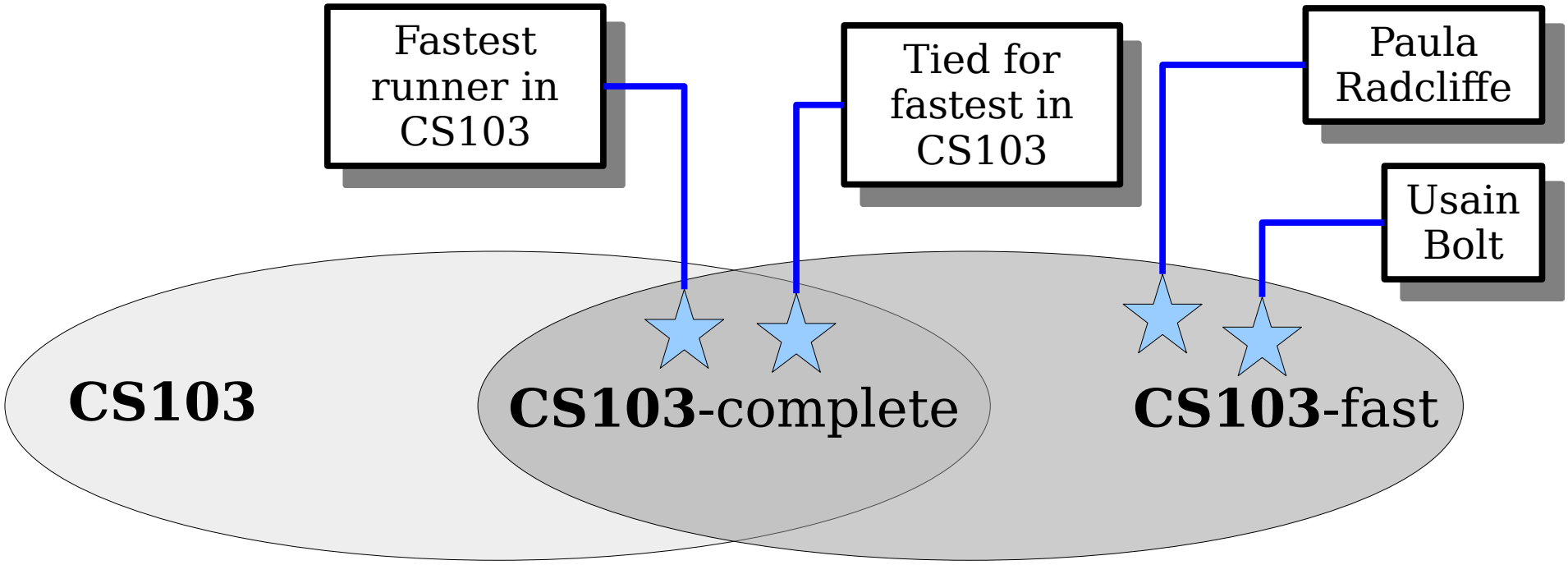
# Preparing for the Exam

- Iris and Stanley will be holding a review session ***Monday, March 17, 3-4 PM*** in CoDa E160!
- We've also released EPP3, a collection of five practice final exams you can use to prepare.
- We've also released the Cumulative Practice Problems list, a gigantic searchable database of problems you can use to brush up on whatever topics you need the most practice with.
- As always, ***keep the TAs in the loop when studying***! That's what we're here for.

Back to CS103!

# **NP**-Hardness and **NP**-Completeness

# ***An Analogy:*** Running Really Fast




For people $A$ and $B$, we say $A \leq_r B$ if $A$'s top running speed is at most $B$'s top speed.

***(Intuitively: B can run at least as fast as A.)***

We say that person $P$ is ***CS103-fast*** if

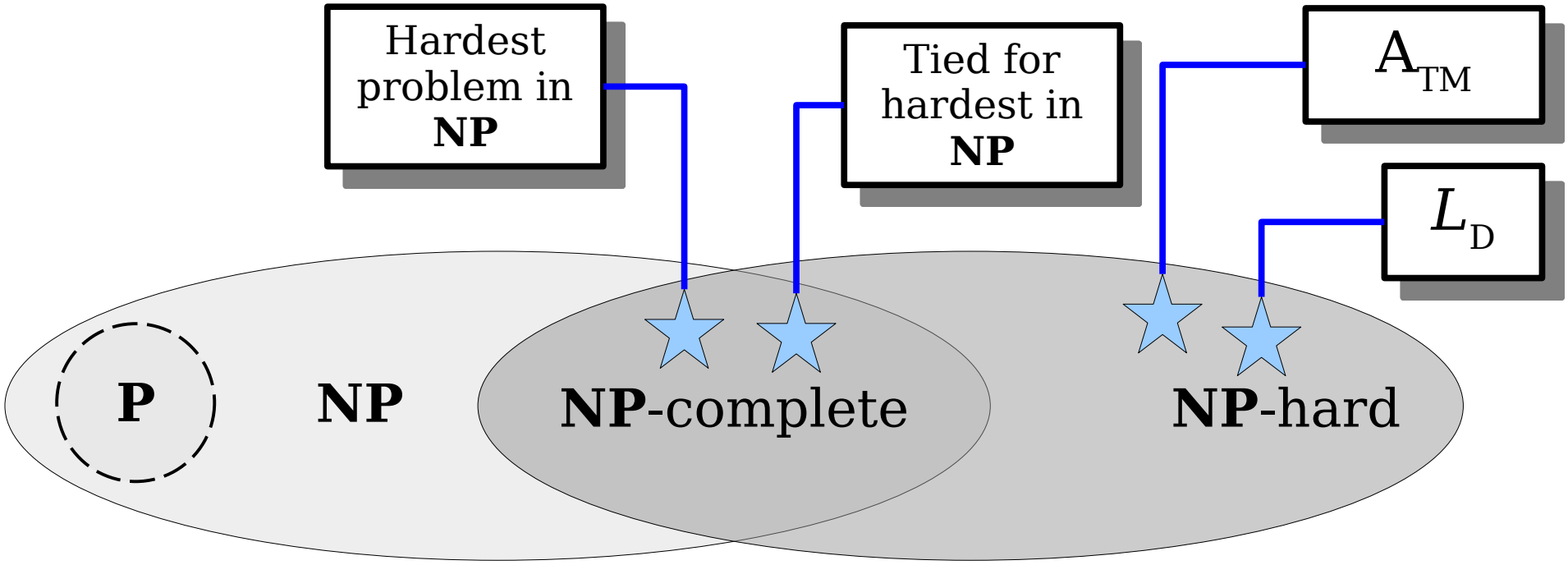
$\forall A \in \mathbf{CS103}.\ A \leq_r P.$

***(How fast are you if you're CS103-fast?)***

We say that person $P$ is ***CS103-complete*** if

$P \in \mathbf{CS103}$ and $P$ is **CS103**-fast.

***(How fast are you if you're CS103-complete?)***




For languages $A$ and $B$, we say $A \leq_p B$ if $A$ reduces to $B$ in polynomial time.

***(Intuitively: B is at least as hard as A.)***

We say that a language $L$ is ***NP-hard*** if

$\forall A \in \mathbf{NP}.\ A \leq_p L.$

***(How hard is a problem that's NP-hard?)***

We say that a language $L$ is ***NP-complete*** if

$L \in \mathbf{NP}$ and $L$ is **NP**-hard.

***(How hard is a problem that's NP-complete?)***

***Intuition:*** The **NP**-complete problems are the hardest problems in **NP**.

If we can determine how hard those problems are, it would tell us a lot about the **P** $\stackrel{?}{=}$ **NP** question.

# The Tantalizing Truth

***Theorem:*** If *any* **NP**-complete language is in **P**, then **P** = **NP**.

***Intuition:*** This means the hardest problems in **NP** aren't actually that hard. We can solve them in polynomial time. So that means we can solve all problems in **NP** in polynomial time.

# The Tantalizing Truth

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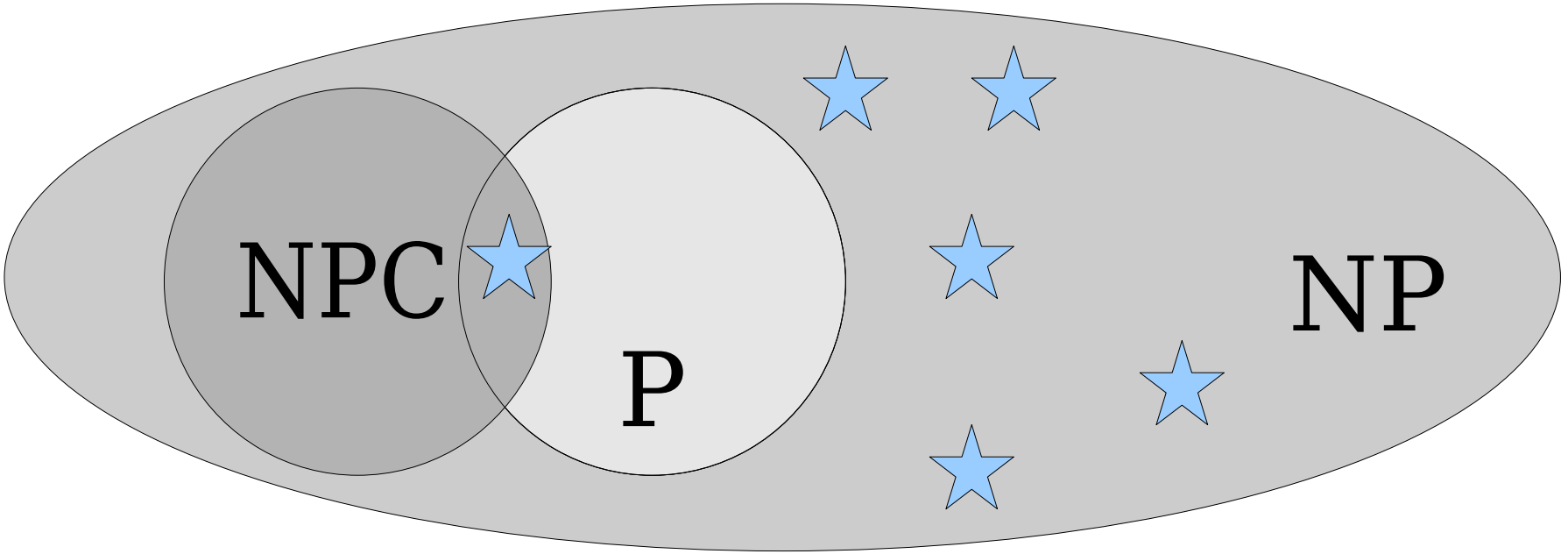

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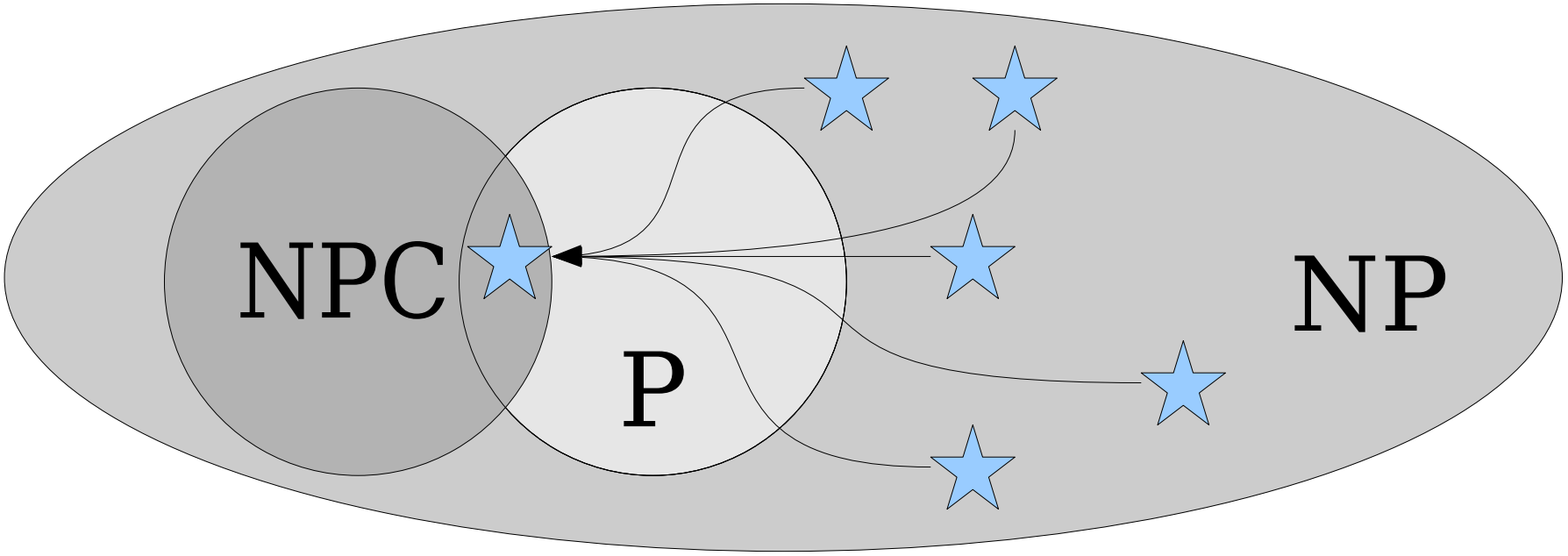

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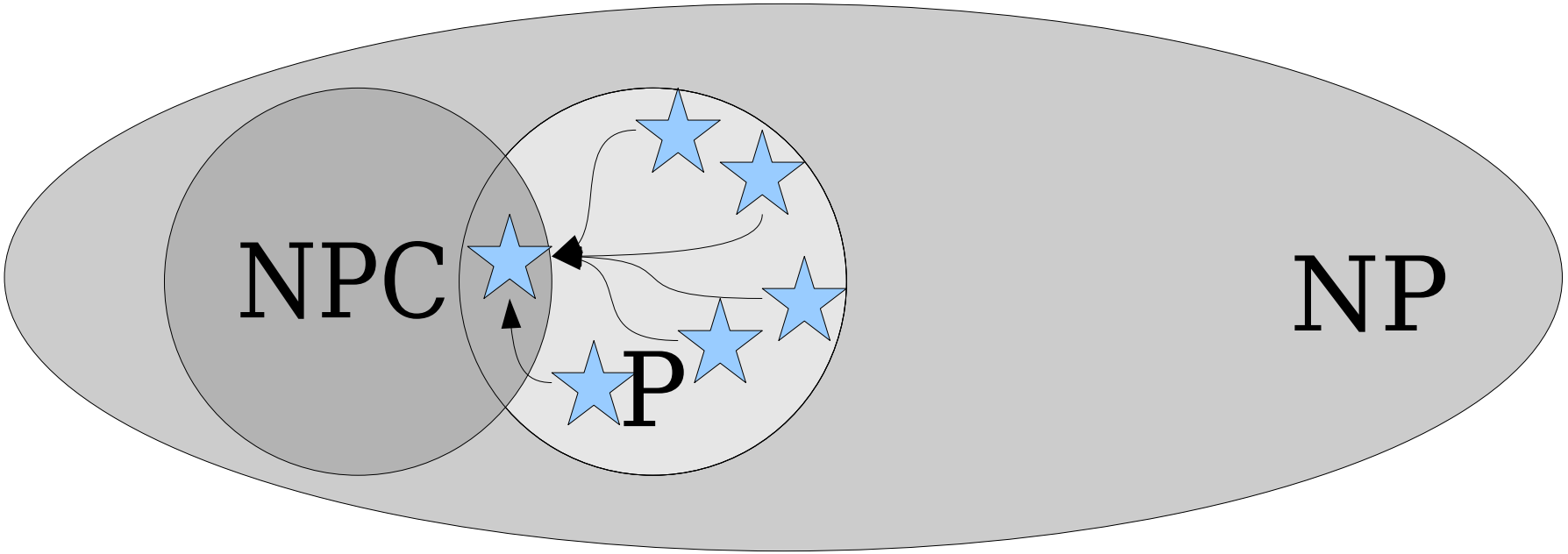

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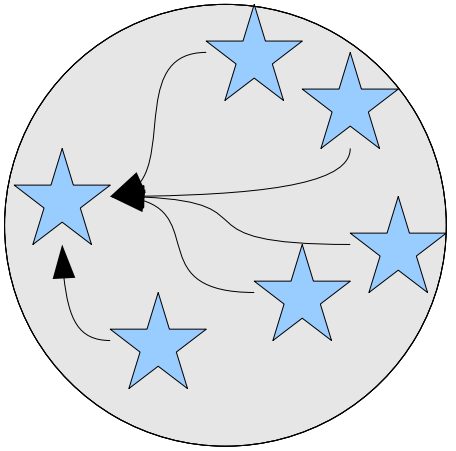
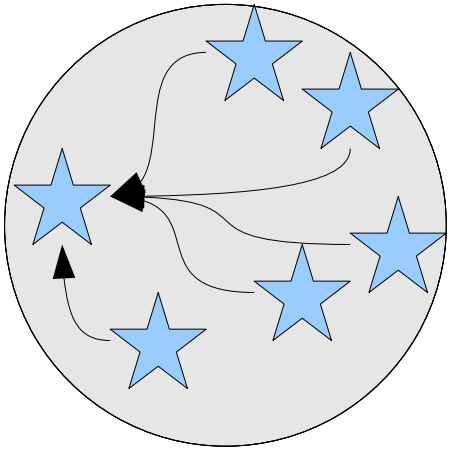


P = NP

# The Tantalizing Truth

***Theorem:*** If *any* **NP**-complete language is in **P**, then **P** = **NP**.

***Proof:*** Suppose that $L$ is **NP**-complete and $L \in \mathbf{P}$. Now consider any arbitrary **NP** problem $A$. Since $L$ is **NP**-complete, we know that $A \leq_p L$. Since $L \in \mathbf{P}$ and $A \leq_p L$, we see that $A \in \mathbf{P}$. Since our choice of $A$ was arbitrary, this means that $\mathbf{NP} \subseteq \mathbf{P}$, so $\mathbf{P} = \mathbf{NP}$. ■



P = NP

# The Tantalizing Truth

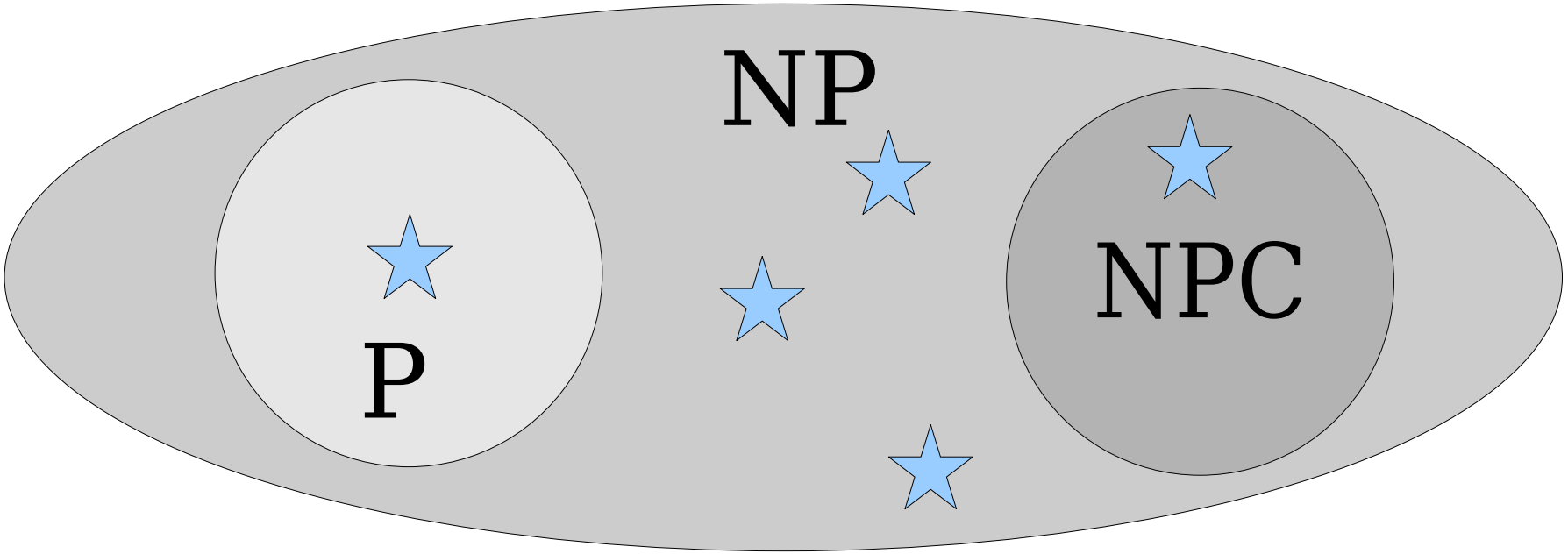
***Theorem:*** If *any* **NP**-complete language is not in **P**, then **P ≠ NP**.

***Intuition:*** This means the hardest problems in **NP** are so hard that they can't be solved in polynomial time. So the hardest problems in **NP** aren't in **P**, meaning **P ≠ NP**.

# The Tantalizing Truth

***Theorem:*** If *any* **NP**-complete language is not in **P**, then **P** ≠ **NP**.

***Proof:*** Suppose that $L$ is an **NP**-complete language not in **P**. Since $L$ is **NP**-complete, we know that $L \in$ **NP**. Therefore, we know that $L \in$ **NP** and $L \notin$ **P**, so **P** ≠ **NP**. ■



***How do we even know NP-complete problems exist in the first place?***

***Theorem (Cook-Levin)***: SAT is **NP**-complete.

***Proof Idea:*** To see that **SAT** ∈ **NP**, show how to make a polynomial-time verifier for it. Key idea: have the certificate be a satisfying assignment.

To show that **SAT** is **NP**-hard, given a polymomial-time verifier $V$ for an arbitrary **NP** language $L$, for any string $w$ you can construct a polynomially-sized formula $\varphi(w)$ that says "there is a certificate $c$ where $V$ accepts $\langle w, c \rangle$." This formula is satisfiable if and only if $w \in L$, so deciding whether the formula is satisfiable decides whether $w$ is in $L$. ■

***Proof:*** Take CS154!

# Why All This Matters

- Resolving $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ is equivalent to just figuring out how hard SAT is.

$$\text{SAT} \in \mathbf{P} \quad \leftrightarrow \quad \mathbf{P = NP}$$

- We've turned a huge, abstract, theoretical problem about solving problems versus checking solutions into the concrete task of seeing how hard one problem is.
- You can get a sense for how little we know about algorithms and computation given that we can't yet answer this question!

# Sample **NP**-Hard Problems

- ***Computational biology:*** Given a set of genomes, what is the most probable evolutionary tree that would give rise to those genomes? *(Maximum parsimony problem)*
- ***Game theory:*** Given an arbitrary perfect-information, finite, two-player game, who wins? *(Generalized geography problem)*
- ***Operations research:*** Given a set of jobs and workers who can perform those tasks in parallel, can you complete all the jobs within some time bound? *(Job scheduling problem)*
- ***Machine learning:*** Given a set of data, find the simplest way of modeling the statistical patterns in that data. *(Bayesian network inference problem)*
- ***Medicine:*** Given a group of people who need kidneys and a group of kidney donors, find the maximum number of people who can receive transplants. *(Cycle cover problem)*
- ***Systems:*** Given a set of processes and a number of processors, find the optimal way to assign those tasks so that they complete as soon as possible. *(Processor scheduling problem)*

# Why All This Matters

- You will almost certainly encounter **NP**-hard problems in practice – they're everywhere!
- If a problem is **NP**-hard, then there is no known algorithm for that problem that
  - is efficient on all inputs,
  - always gives back the right answer, and
  - runs deterministically.
- ***Useful intuition:*** If you need to solve an **NP**-hard problem, you will either need to settle for an approximate answer, an answer that's likely but not necessarily right, or have to work on really small inputs.

# Next Time

- ***Why All This Matters***
- ***Where to Go from Here***
- ***A Final "Your Questions"***
- ***Parting Words!***